COSC 6222 :: Test #2

Attempt all questions. All questions have equal weight.

Take home exam: Due Monday, April 26, 11:59 PM. Submit by email to: `aeckford@yorku.ca`
Consulting with others is prohibited. Otherwise, any reference material is permitted.
You must declare all references you consulted.

1. A channel with *multiplicative noise* has inputs $X \in \{+1, -1\}$, noise $Z \in \{0, 1, a\}$ for some known real number a, and outputs Y, related by

$$Y = ZX.$$

   Let $\Pr(X = +1) = \Pr(X = -1) = 1/2$, and let $\Pr(Z = 0) = \Pr(Z = 1) = \Pr(Z = a) = 1/3$. Find $I(X; Y)$ as a function of a.

2. Suppose you have a telephone keypad with 10 buttons, corresponding to the numbers 0-9.

   a. If pushing a button transmits the associated letter, and the receiver observes the transmission perfectly, what is the capacity of the channel?

   b. Suppose the keypad contains errors, so that pressing $k \in \{0, 1, \ldots, 9\}$ transmits either k or $(k + 1) \mod 10$. For each key press, the correct transmission occurs with probability $(1 - p)$, and the erroneous transmission occurs with probability p, where $0 < p < 1$. For example, if 0 is pressed, 0 is transmitted with probability $(1 - p)$, and 1 is transmitted with probability p. Find the capacity of the resulting channel.

   c. Propose a transmission strategy so that keypresses are always transmitted without error. For what values of p does this achieve the capacity of the channel that you obtained in part b?

3. Let $\mathbf{X} = [X_1, X_2, \ldots, X_k] \in \{0, 1\}^k$ represent a vector of channel inputs, and let $\mathbf{Y} = [Y_1, Y_2, \ldots, Y_k] \in \{0, 1\}^k$ represent the corresponding vector of channel outputs. Furthermore, $p(\mathbf{Y}|\mathbf{X}) = \prod_{i=1}^{k} p(Y_i|X_i)$, where the channel is a binary symmetric channel (BSC) for each i, i.e.,

$$p(Y_i|X_i) = \begin{cases} 1 - p_i, & Y_i = X_i, \\ p_i, & Y_i \neq X_i. \end{cases}$$

   The values of p_i are known in advance, and $p_i \neq p_j$ in general for $i \neq j$.

   a. Show that the capacity-achieving input distribution $p(x)$ is given by $p(x) = \prod_{i=1}^{k} p(x_i)$, where $p(x_i = 1) = p(x_i = 0) = 1/2$.

   b. Using the input distribution from part a, calculate $I(\mathbf{X}; \mathbf{Y})$.

4. Let X be a channel input, and let Y be a channel output. Let $f(\cdot)$ represent any function of Y.

   a. Show that $I(X; f(Y)) \leq I(X; Y)$ for any $f(\cdot)$.

   b. Under what circumstances does $I(X; f(Y)) = I(X; Y)$?