ENG2200
Electric Circuits

Chapter 10

Laplace Transform

Objectives

- Be able to calculate Laplace and inverse Laplace transform for any function.
- Be able to transform a circuit into the *s* domain
- Know hoe to analyze circuits in the *s* domain

Laplace Transform

- The idea of any transform is to change from domain *A* to domain *B*. Solve the problem in domain *B* and return again to domain *A*.
- For example, the phasor domain
- A sinusoidal wave could be represented in the time and phasor domain
- It is easier to solve the problem in the phasot domain

Laplace Transform

$$Laplace\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Laplace Transform

- The Laplace transform is defined as

$$Laplace\{f(t)\} = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

- This is the **one-sided** or **unilateral Laplace Transform**
- The lower limit is 0^- What happens before that is included in the initial conditions

Some Important Functions

- The step function u(t)
- The impulse function δ(t)

Laplace Transform Pairs

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$1/s$
t	$1/s^2$
e^{-at}	$1/(s+a)$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
te^{-at}	$\dfrac{1}{(s+a)^2}$

Laplace Transform Properties

Operation	$f(t)$	$F(s)$
Multiplication	$Kf(t)$	$KF(s)$
addition	$f_1(t)+f_2(t)$	$F_1(s)+F_2(s)$
derivative(t)	$\dfrac{df(t)}{dt}$	$sF(s)-f(0^-)$
2nd derivative(t)	$\dfrac{d^2f(t)}{dt^2}$	$s^2F(s)-sf(0^-)-\dfrac{df(0^-)}{dt}$
Integral(t)	$\int_0^t f(x)dx$	$\dfrac{F(s)}{s}$
Translation (t)	$f(t-a)u(t-a), a>0$	$e^{-as}F(s)$
Translation (s)	$e^{-at}f(t)$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

Laplace Transform Properties

Operation	$f(t)$	$F(s)$
Scaling	$f(at), a>0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
derivative(s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative(s)	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
integral(s)	$\frac{f(t)}{t}$	$\int_s^{\infty} F(u)du$

Inverse Laplace Transform—Partial Fraction

- Our focus is on proper rational fraction m≤n

$$F(s)=\frac{a_m s^m + a_{m-1}s^{m-1}+\ldots+a_1 s+a_0}{s^n+b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}$$

$$F(s)=\frac{a_m s^m + a_{m-1}s^{m-1}+\ldots+a_1 s+a_0}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

- Distinct poles
- Repeated poles
- Complex poles

Distinct Poles

- If all the poles are distinct

$$F(s) = K + \frac{A_1}{(s-p_1)} + \frac{A_2}{(s-p_2)} + \ldots + \frac{A_n}{(s-p_n)}$$

$$K = \lim_{s\to\infty} F(S)$$

$$K = \lim_{s\to p_i}\left[(s-p_i)F(S)\right]$$

$$f(t) = K\delta(t) + A_1 e^{p_1 t}u(t) + A_2 e^{p_2 t}u(t) + .. + A_n e^{p_n t}u(t)$$

Repeated Poles

- One or more poles are repeated

$$F(s) = \frac{n(s)}{(s-a)^k d(s)}$$

$$F(s) = \frac{A_1}{(s-a)} + \frac{A_2}{(s-a)^2} + \ldots + \frac{A_n}{(s-a)^k} + \frac{n(s)}{d(s)}$$

$$A_k = (s-a)^k F(s)\Big|_{s=a} = \left.\frac{n(s)}{d(s)}\right|_{s=a}$$

$$A_{k-1} = \frac{d}{ds}[(s-a)^k F(s)] = \left.\frac{d}{ds}\left[\frac{n(s)}{d(s)}\right]\right|_{s=a}$$

$$A_{k-i} = \frac{1}{i!}\frac{d^i}{ds^i}[(s-a)^k F(s)] = \left.\frac{1}{i!}\frac{d^i}{ds^i}\left[\frac{n(s)}{d(s)}\right]\right|_{s=a}$$

Distinct Complex Poles

- Complex poles

$$F(s)=\frac{n(s)}{\left[(s+a)^2+\omega^2\right]d(s)}=\frac{n(s)}{(s+a+j\omega)(s+a-j\omega)d(s)}$$

$$F(s)=\frac{A+jB}{(s+a+j\omega)}+\frac{A-jB}{(s+a+j\omega)}+\frac{n(s)}{d(s)}$$

$$A+jB=\left[(s+a+jw)F(s)\right]_{s=-a-jw}$$

Laplace Transform --Applications

- The v-I relationship of a capacitor

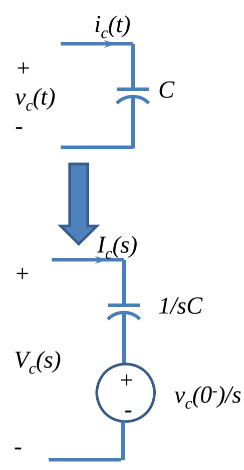
$$v_c(t)=\frac{1}{c}\int_{-\infty}^{t} i_c(\tau)d\tau$$

$$L\{v_c(t)\}=L\left\{\frac{1}{C}\int_{-\infty}^{t} i_c(\tau)d\tau\right\}$$

$$V_c(s)=\frac{1}{Cs}I_c(s)+\frac{1}{Cs}\int_{-\infty}^{0-} i_c(\tau)d\tau$$

$$V_c(s)=\frac{1}{Cs}I_c(s)+\frac{1}{s}v_c(0^-)$$

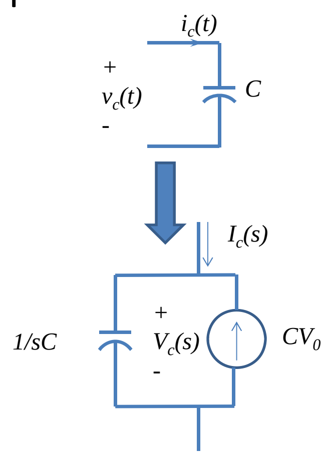
$$V_c=\frac{I}{sC}+\frac{V_0}{s}$$



Laplace Transform --Applications

- The v-I relationship of a capacitor

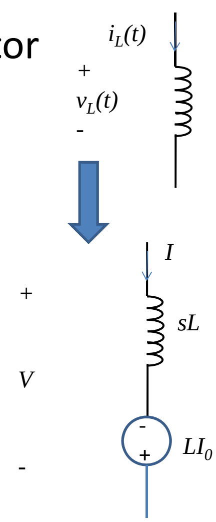
$$V_c(s) = \frac{1}{Cs} I_c(s) + \frac{1}{s} v_c(0^-)$$
$$\frac{1}{Cs} I_c(s) = V_c(s) - \frac{1}{s} v_c(0^-)$$
$$I_c(s) = sCV_c(s) - Cv_c(0^-)$$
$$I = sCV - CV_0$$



Laplace Transform --Applications

- The v-I relationship of a inductor

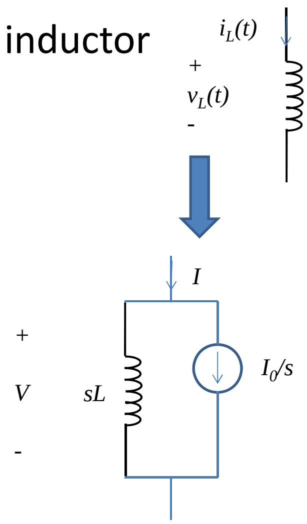
$$v_c(t) = L\frac{di}{dt}$$
$$V(s) = L\left[sI(s) - i(0^0)\right]$$
$$V = sLI - LI_0$$



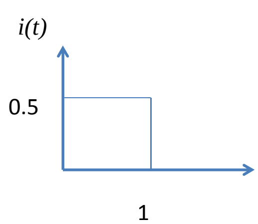
Laplace Transform --Applications

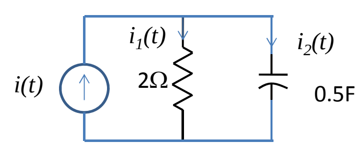
- The v-I relationship of a inductor

$$V = sLI - LI_0$$
$$sLI = V + LI_0$$
$$I = \frac{V}{sL} + \frac{I_0}{s}$$



Example





Example