

ENG2200

Electric Circuits

Chapter 8
RLC Circuit
Natural and Step Response

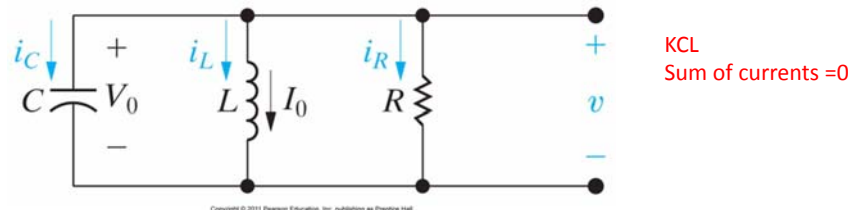
ENG2200 Topics to be covered

- Be able to determine the natural response of RLC circuits
- Be able to determine the step response of RLC circuits.

RLC Circuits

- The first step is to write either KVL or KCL for the circuit.
- Take the derivative to remove any integration
- Solve the resulting differential equation

Parallel RLC circuit



$$i_R + i_L + i_C = 0$$

$$\frac{v}{R} + \frac{1}{L} \int_0^t v d\tau + I_0 + C \frac{dv}{dt} = 0$$

Differentiating with respect to t, we get

$$C \frac{d^2v}{dt^2} + \frac{1}{R} \frac{dv}{dt} + \frac{v}{L} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{v}{LC} = 0 \quad \leftarrow \text{Solve this equation}$$

Parallel RLC Circuits

- How to solve this differential equation?
- We can not separate the variables like we did with the RC or RL circuits.
- Without going into a lot of Math, we claim the solution will be in the form $v = Ae^{st}$
 - Exponential is the only function where high order of the derivatives have the same form (exponential)
 - First order (RL or RC) have the same form

Parallel RLC Circuits

- Assuming $v = Ae^{st}$ and substituting in the equation

$$\frac{d^2v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{v}{LC} = 0$$

$$v = Ae^{st}$$

$$As^2e^{st} + \frac{As}{RC}e^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st} \left(s^2 + \frac{s}{RC} + \frac{1}{LC} \right) = 0$$

Either A=0 (trivial solution) or
The quadratic part is 0

Parallel RLC Circuits

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Both of these 2 values satisfy the equation, their sum does.

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

With a special case when $s_1 = s_2$

Parallel RLC Circuits

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

Neper frequency

Resonant radian frequency

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi\omega_0 \pm \omega_0 \sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \xi = \frac{1}{2R} \sqrt{\frac{L}{C}}$$

Damping ratio

Parallel RLC Circuits

- The solution of the differential equation depends on the values of s_1 and s_2
- For simplicity assume A_1 and A_2 to be 1

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- Consider three cases

Overdamping $\alpha > \omega_0$, $\zeta > 1$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

$$s = -\alpha \pm j\omega_d \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$v(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_1 e^{-\alpha t} \sin(\omega_d t)$$

Critically Damped $\alpha=\omega_0$, $\zeta=1$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

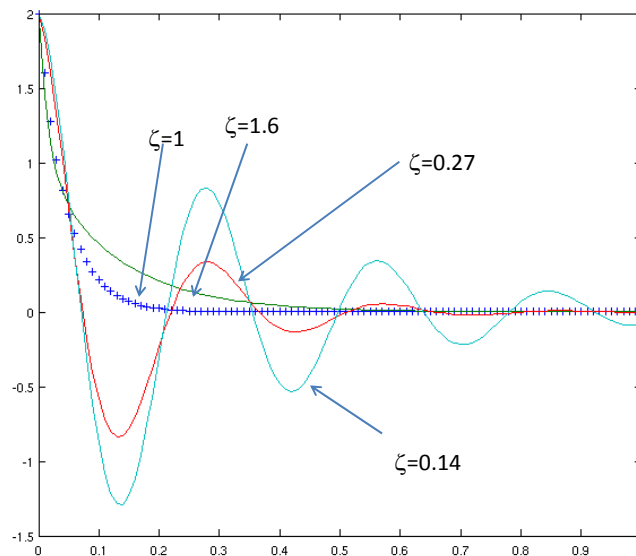
$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

$$s_1 = s_2 = -\alpha$$

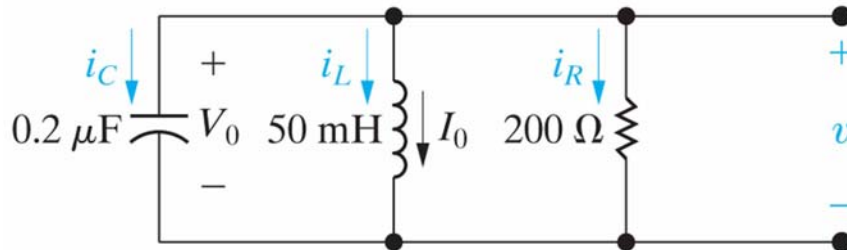
$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$



$$A_1=A_2=B_1=B_2=D_1=D_2=1$$

Parallel RLC Circuits

- Now we have a solution for the problem with



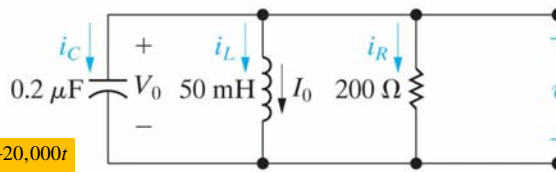
Copyright © 2011 Pearson Education, Inc. publishing as Prentice Hall

instantaneously

$$v_C(0^-) = v_C(0) = v_C(0^+)$$

$$i_L(0^-) = i_L(0) = i_L(0^+)$$

Example



Copyright © 2011 Pearson Education, Inc. publishing as Prentice Hall

$$v(t) = A_1 e^{-5,000t} + A_2 e^{-20,000t}$$

$$v(0^+) = 12 = A_1 + A_2$$

Since the initial voltage across C (and R) is 12, the initial current in R
 $= 12/200 = 60 \text{ mA}$
 $i_R(0^+) = 60 \text{ mA}$

$$i_C(0^+) = -i_R(0^+) - i_L(0^+) = -90 \text{ mA}$$

$$i_C(0^+) = C \frac{dv(0^+)}{dt}$$

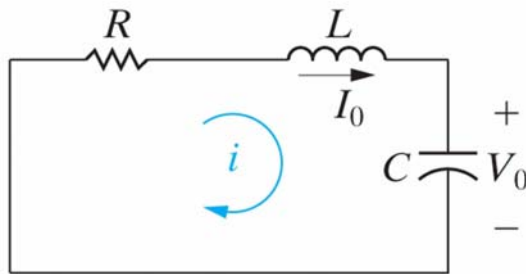
$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{-90 \text{ mA}}{0.2 \mu\text{F}} = -450,000 \text{ V/s}$$

$$\frac{dv(0^+)}{dt} = -450,000 \text{ V/s} = -5000A_1 - 20,000A_2$$

2 equation in 2
 unknowns, solve to get
 A_1 and A_2

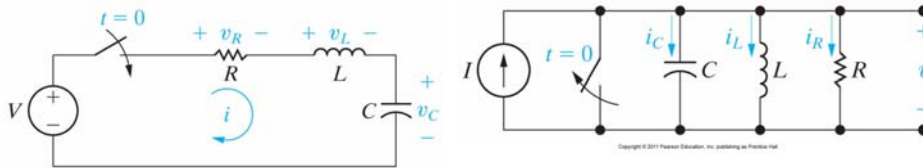
Series RLC

- Solved in the lab manual
- Only difference is $\alpha = R/2L$



Copyright © 2011 Pearson Education, Inc. publishing as Pearson Hall

Step Response of RLC Circuit



Copyright © 2011 Pearson Education, Inc. publishing as Pearson Hall

$$\begin{aligned}
 v_L + v_C + v_R &= V \\
 L \frac{di}{dt} + v_C + iR &= V \\
 i &= C \frac{dv_C}{dt} \\
 LC \frac{d^2 v_C}{dt^2} + v_C + RC \frac{dv_C}{dt} &= V \\
 \frac{d^2 v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{v_C}{LC} &= \frac{V}{LC}
 \end{aligned}$$

$$\begin{aligned}
 i_L + i_C + i_R &= I \\
 i_L + C \frac{dv}{dt} + \frac{v}{R} &= I \\
 v &= L \frac{di_L}{dt} \\
 LC \frac{d^2 i_L}{dt^2} + \frac{L}{R} \frac{di_L}{dt} + i_L &= I \\
 \frac{d^2 i_L}{dt^2} + \frac{1}{RC} \frac{di_L}{dt} + \frac{i_L}{LC} &= \frac{I}{LC}
 \end{aligned}$$

Step Response of RLC Circuits

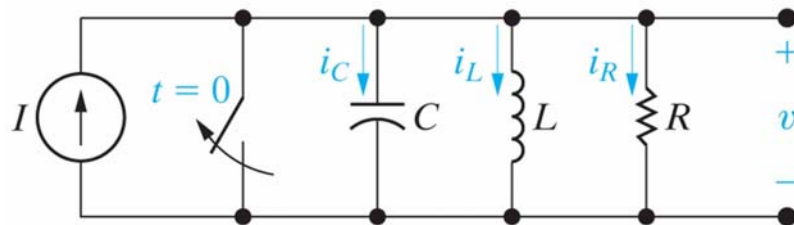
- A topic for a course in Math.
- Generally speaking, the solution of a second-order DE with a constant driving force equals the forced response plus the a response function identical to the natural response.

$$i = I_f + \left\{ \begin{array}{l} \text{function of the same form} \\ \text{as natural response} \end{array} \right\}$$

$$v = V_f + \left\{ \begin{array}{l} \text{function of the same form} \\ \text{as natural response} \end{array} \right\}$$

Example

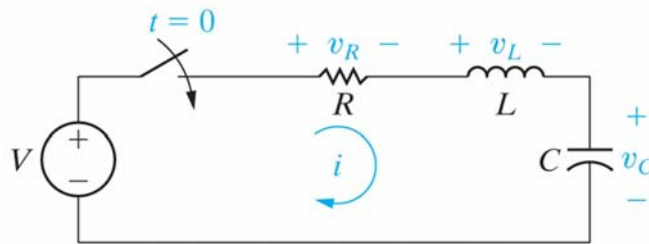
- What is the final I_f



Copyright © 2011 Pearson Education, Inc. publishing as Prentice Hall

Example

- What is the final V_f



Copyright © 2011 Pearson Education, Inc. publishing as Prentice Hall