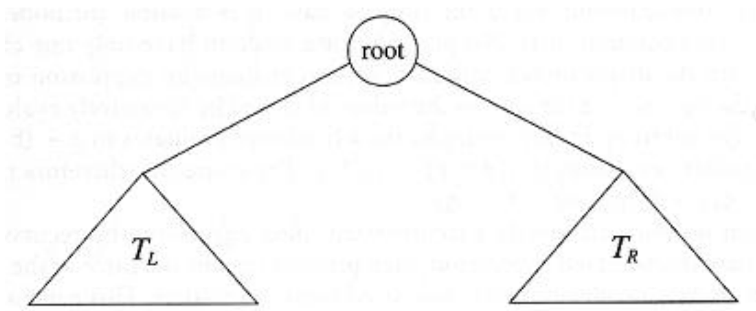
Binary Trees

Section 7.3

CSE 2011

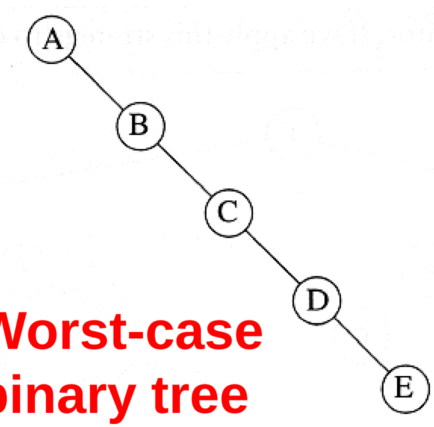
Binary Trees

- A tree in which each node can have at most two children.



Generic binary tree

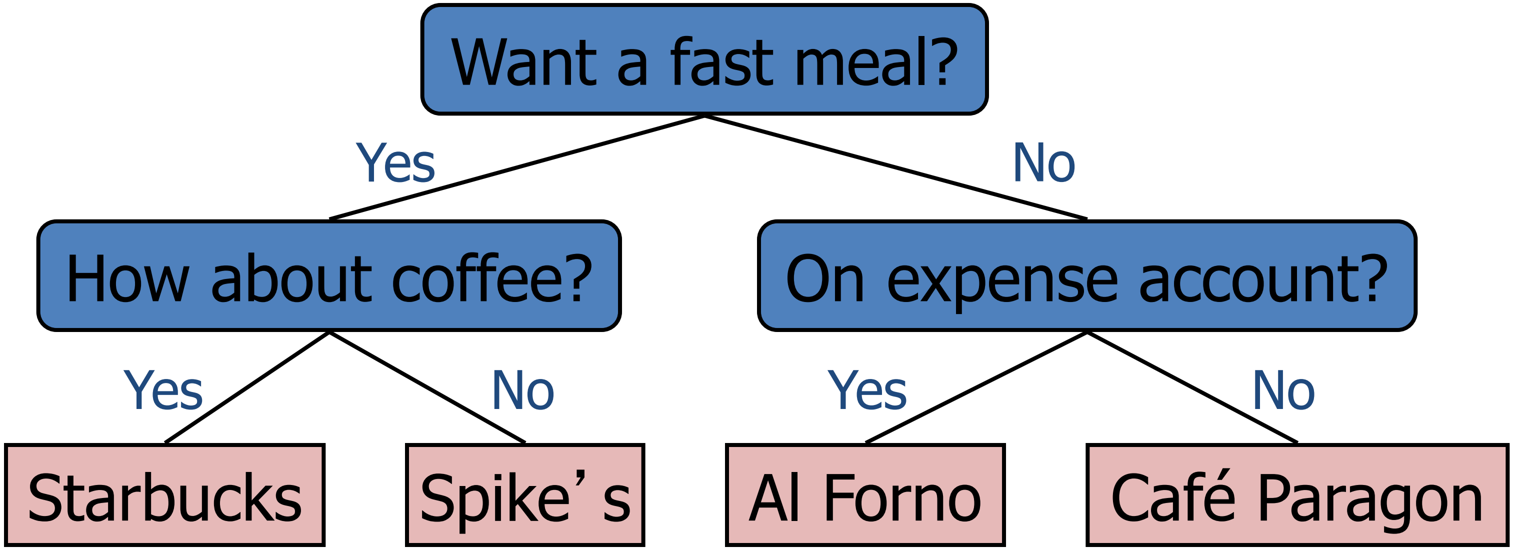
- The depth of an "average" binary tree is considerably smaller than N In the worst case, the depth can be as large as N – 1.



Worst-case binary tree

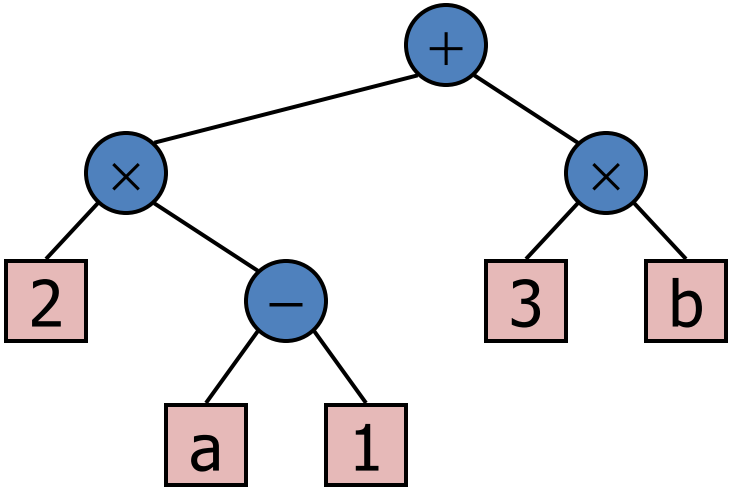
Decision Tree

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision



Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands
- Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$



Tree ADT (review)

- We use positions to abstract nodes (position ≡ node)
- Generic methods:
  - integer size()
  - boolean isEmpty()
  - Iterator elements()
  - positionIterator positions()
- Accessor methods:
  - position root()
  - position parent(p)
  - positionIterator children(p)

- Query methods:
  - boolean isInternal(p)
  - boolean isExternal(p)
  - boolean isRoot(p)
- Update method:
  - object replace (p, e): replace with e and return element stored at node p
- Additional update methods may be defined by data structures implementing the Tree ADT

BinaryTree ADT

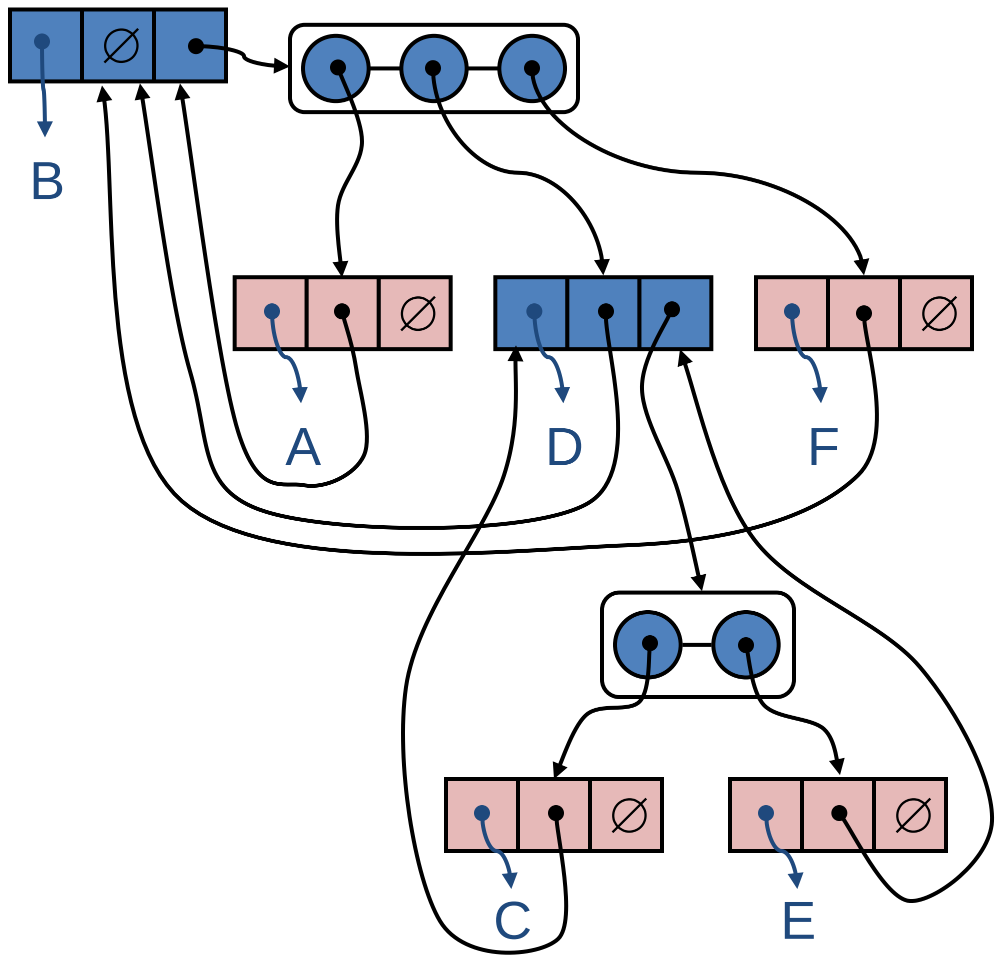
- The BinaryTree ADT extends the Tree ADT, i.e., it inherits all the methods of the Tree ADT
- Additional methods:
  - position left(p)
  - position right(p)
  - boolean hasLeft(p)
  - boolean hasRight(p)
- Update methods may be defined by data structures implementing the BinaryTree ADT

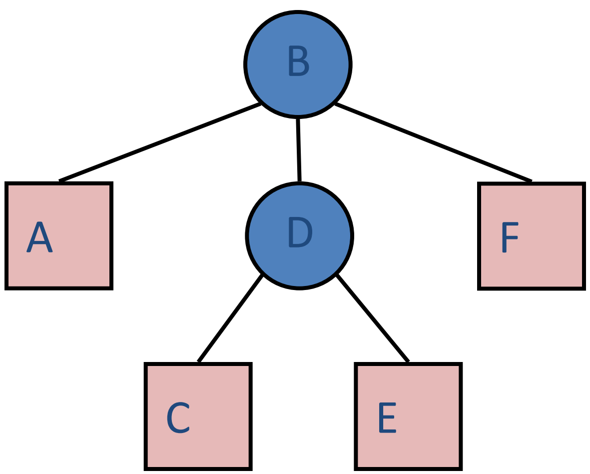
Implementing Binary Trees

- Arrays?
  - Discussed later
- Linked structure?

Linked Structure for Trees (review)

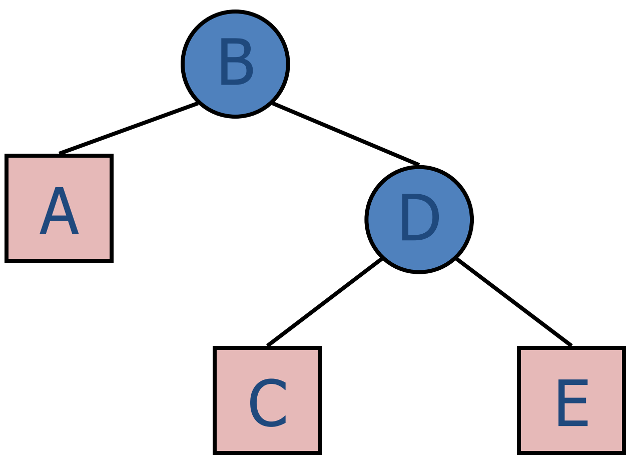
- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes

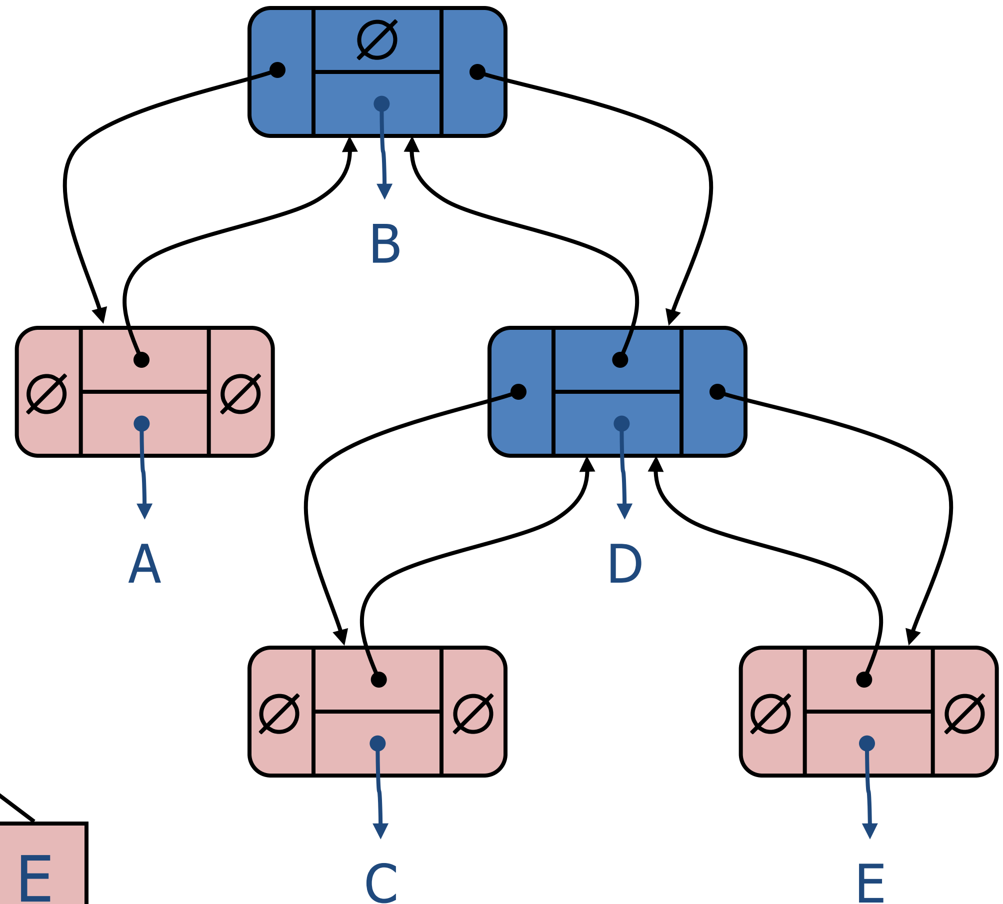




Linked Structure of Binary Trees (2)

- A node is represented by an object storing
  - Element
  - Parent node
  - Left child node
  - Right child node





Linked Structure of Binary Trees

```
class BinaryNode {
   Object          element
   BinaryNode      left;
   BinaryNode      right;
   BinaryNode      parent;
}
```

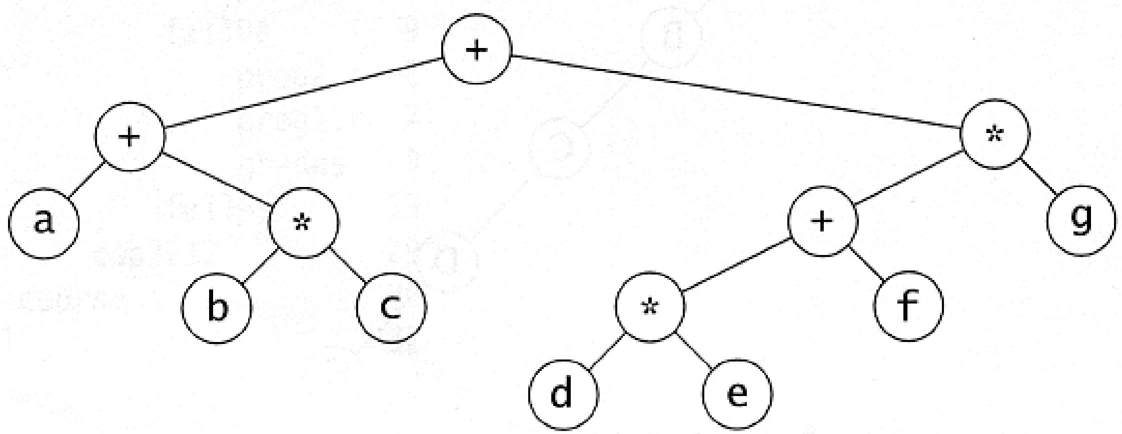


Figure 4.14 Expression tree for (a + b * c) + ((d * e + f) * g)

Binary Tree Traversal

- Preorder (node, left, right)
- Postorder (left, right, node)
- Inorder (left, node, right)

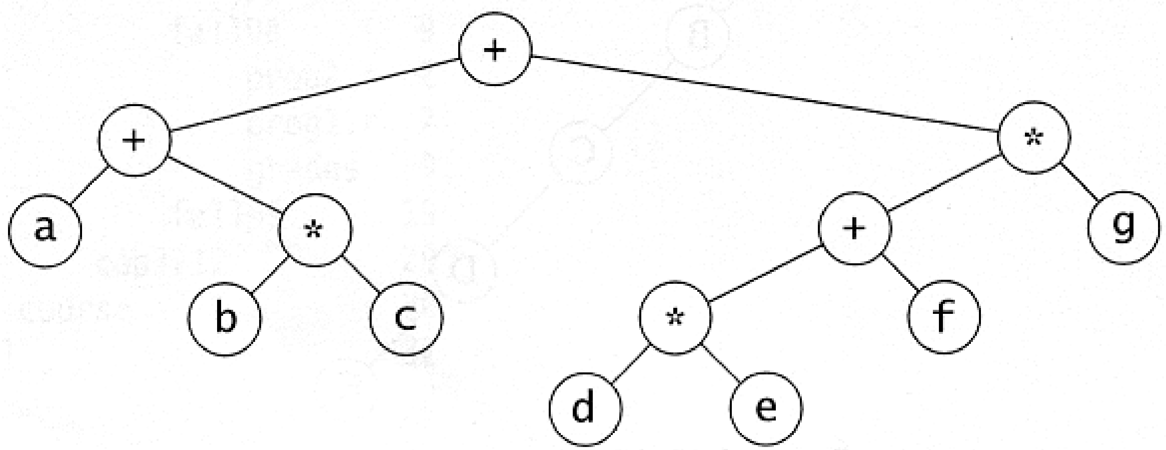


Figure 4.14 Expression tree for (a + b * c) + ((d * e + f) * g)

Preorder Traversal: Example

- Preorder traversal
  - node, left, right
  - prefix expression
    - + + a * b c * + * d e f g

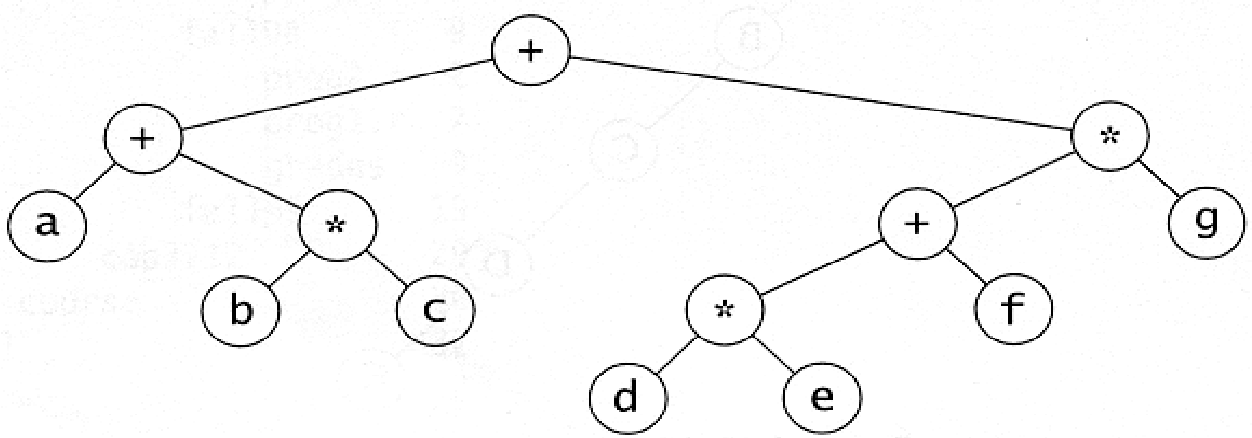


Figure 4.14 Expression tree for (a + b * c) + ((d * e + f) * g)

Postorder Traversal: Example

- Postorder traversal
  - left, right, node
  - postfix expression
    - a b c * + d e * f + g * +

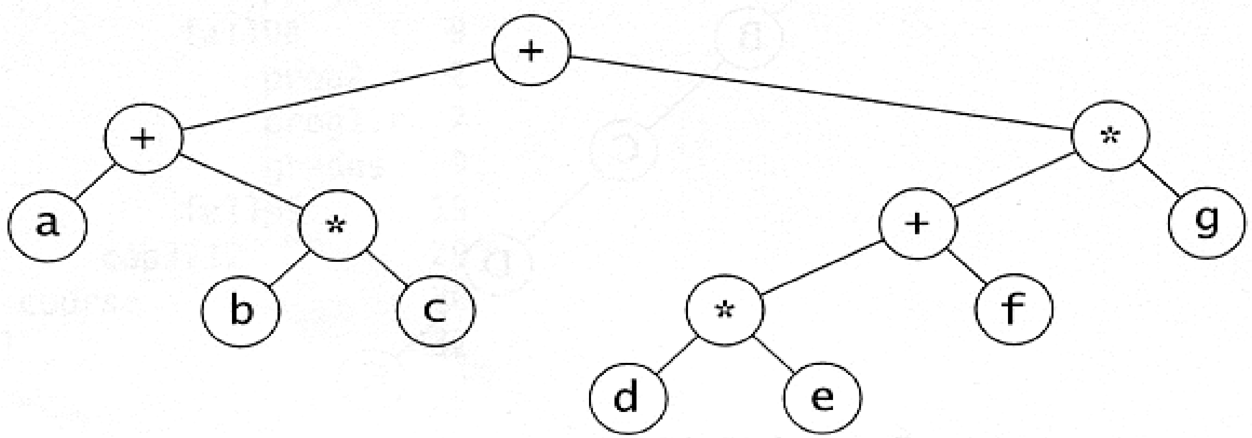


Figure 4.14 Expression tree for (a + b * c) + ((d * e + f) * g)

Inorder Traversal: Example

- Inorder traversal
  - left, node, right
  - infix expression
    - a + b * c + d * e + f * g

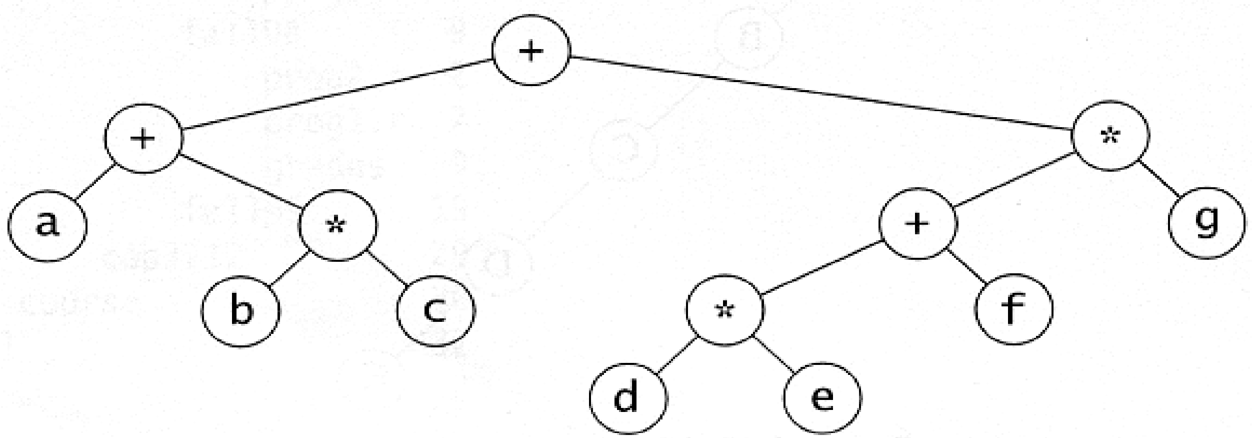


Figure 4.14 Expression tree for (a + b * c) + ((d * e + f) * g)

Pseudo-code for Binary Tree Traversal

```
Algorithm Preorder(x)
Input: x is the root of a subtree.
1.   if x ≠ NULL
2.      then output key(x);
3.           Preorder(left(x));
4.           Preorder(right(x));
```

```
Algorithm Inorder(x)
Input: x is the root of a subtree.
1.   if x ≠ NULL
2.      then Inorder(left(x));
3.           output key(x);
4.           Inorder(right(x));
```

```
Algorithm Postorder(x)
Input: x is the root of a subtree.
1.   if x ≠ NULL
2.      then Postorder(left(x));
3.           Postorder(right(x));
4.           output key(x);
```

Properties of Proper Binary Trees

- A binary trees is ***proper*** if each node has either zero or two children.
- Level: depth

  The root is at level 0

  Level d has at most 2^d nodes
- Notation:

  n number of nodes

  e number of external (leaf) nodes

  i number of internal nodes

  h height

$$n = e + i$$

$$e = i + 1$$

$$h + 1 \le e \le 2^h$$

$$n = 2e - 1$$

$$h \le i \le 2^h - 1$$

$$2h + 1 \le n \le 2^{h+1} - 1$$

$$\log_2 e \le h \le e - 1$$

$$\log_2 (i + 1) \le h \le i$$

$$\log_2 (n + 1) - 1 \le h \le (n - 1)/2$$

Properties of (General) Binary Trees

- Level: depth
  The root is at level 0
  Level *d* has at most $2^{\boldsymbol{d}}$ nodes
- Notation:
  $\boldsymbol{n}$ number of nodes
  $\boldsymbol{e}$ number of external (leaf) nodes
  $\boldsymbol{i}$ number of internal nodes
  $\boldsymbol{h}$ height

$$\boldsymbol{h}+1 \leq \boldsymbol{n} \leq 2^{\boldsymbol{h+1}} - 1$$

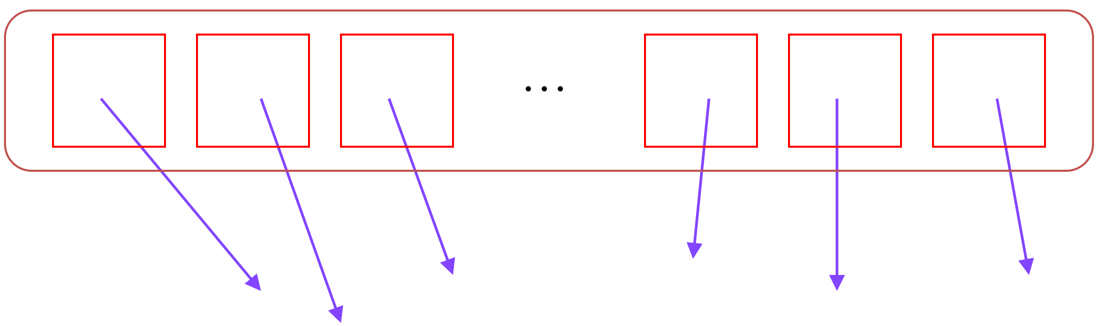
$$1 \leq \boldsymbol{e} \leq 2^{\boldsymbol{h}}$$

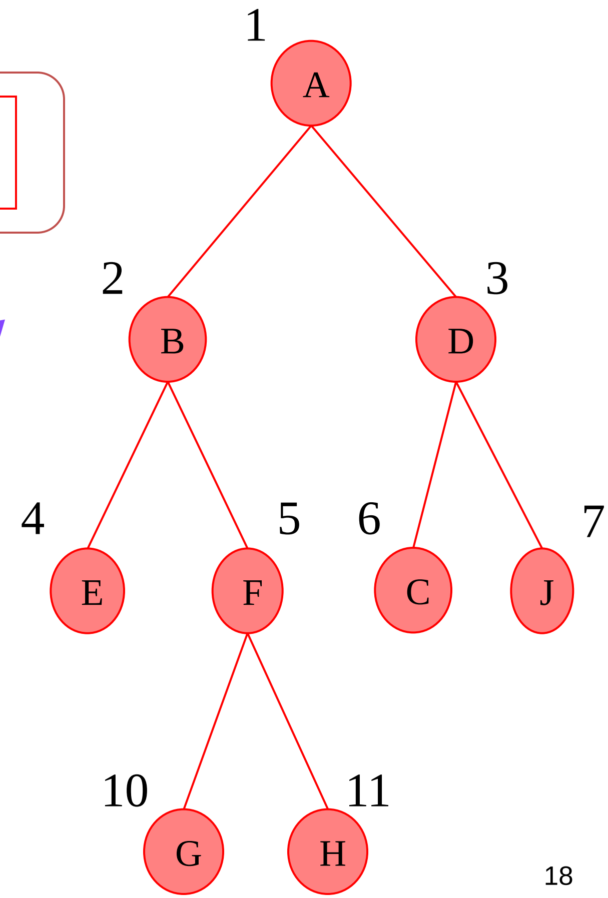
$$\boldsymbol{h} \leq \boldsymbol{i} \leq 2^{\boldsymbol{h}} - 1$$

$$\log_2(\boldsymbol{n}+1) - 1 \leq \boldsymbol{h} \leq \boldsymbol{n} - 1$$

Array-Based Implementation

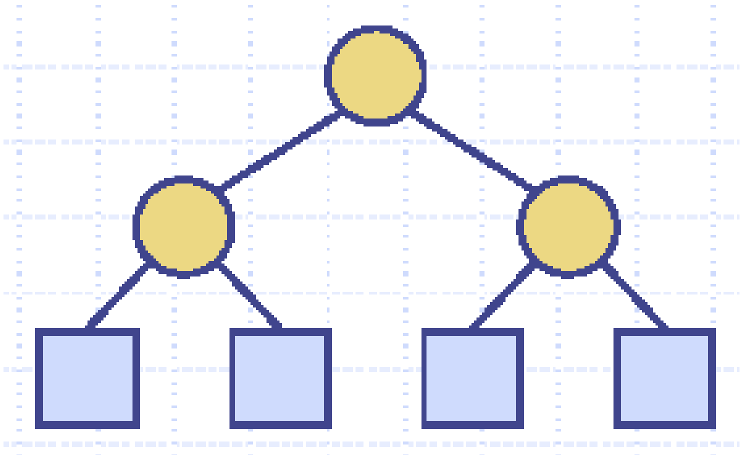
- Nodes are stored in an array.

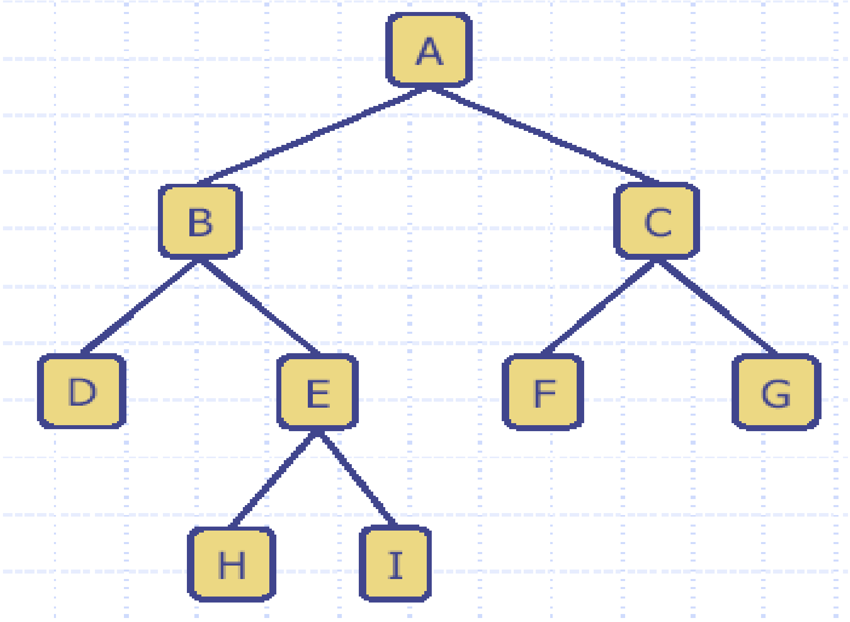




- Let rank(v) be defined as follows:
  - rank(root) = 1
  - if v is the left child of parent(v),
    rank(v) = 2 * rank(parent(v))
  - if v is the right child of parent(v),
    rank(v) = 2 * rank(parent(v)) + 1

Array Implementation of Binary Trees





Each node v is stored at index i defined as follows:

- If v is the root, $i = 1$
- The left child of v is in position $2i$
- The right child of v is in position $2i + 1$
- The parent of v is in position ???

Space Analysis of Array Implementation

- n: number of nodes of binary tree T
- p_M: index of the rightmost leaf of the corresponding ***full*** binary tree (or size of the full tree)
- N: size of the array needed for storing T; $N = p_M + 1$

Best-case scenario: balanced, full binary tree $p_M = n$

Worst case scenario: unbalanced tree

- Height $h = n – 1$
- Size of the corresponding full tree:

  $p_M = 2^{h+1} – 1 = 2^n – 1$
- $N = 2^n$

Space usage: $O(2^n)$

Arrays versus Linked Structure

Linked structure

- Slower operations due to pointer manipulations
- Use less space if the tree is unbalanced
- AVL trees: rotation (restructuring) code is simple

Arrays

- Faster operations
- Use less space if the tree is balanced (no pointers)
- AVL trees: rotation (restructuring) code is complex

Next lecture …

- Binary Search Trees (10.1)
- AVL Trees (10.2)