Random Signals

Why Random Signals?

- All useful message signal appear random
- Two types of imperfections in a communication channel:
  - Deterministic imperfection, such as linear and nonlinear distortions, inter-symbol interference, etc.
  - Nondeterministic imperfection, such as addition of noise, interference, multipath fading, etc.
- We are concerned with the methods used to describe and characterize a random signal, generally referred to as a random process.

Random Variables (1)

1. Random experiment: its outcome, for some reason, cannot be predicted with certainty

   Examples: throwing a die, tossing a coin, and measuring a electrical component

2 Sample Space: is a set of all possible outcomes

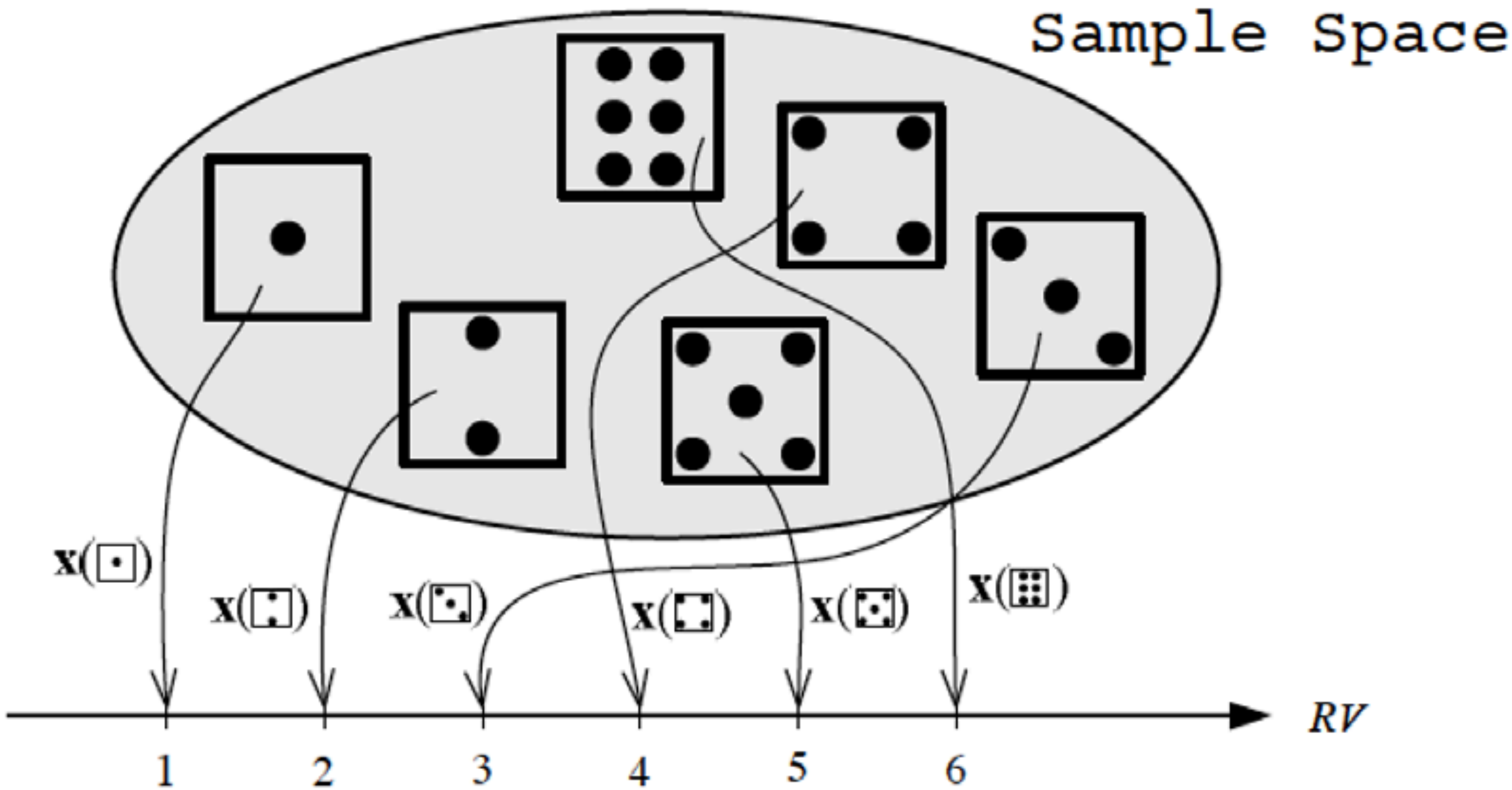
   Example I: $S = \{HH, HT, TH, TT\}$ in tossing of a coin twice.

   Example II: $S = \{NNN, NND, NDN, NDD, DNN, DND, DDN, DDD\}$ in testing three electronic components with N denoting nondefective and D denoting defective.

Random Variables (2)

3. Random variable is a function that associates a real number with each outcome of an experiment.

   Example of throwing a die



Random Variables (2)

3. Random variable examples

   Example I: In tossing of a coin, we count the number of heads and call it the RV X

   Possible values of X: 0, 1, 2.

   Example II: In testing of electronic components, we associate RV Y to the number of defective components. Possible values of Y: 0, 1, 2, 3.

4. Discrete RV: takes discrete set of values. RV X and Y in above examples are discrete

5. Continuous RV: takes values on an analog scale.

   Example III: Distance traveled by a car in 5 hours

   Example IV: Measured voltage across a resistor using an analog meter.

Random Variables (3)

6. Probability density function of a discrete RV: is the distribution of probabilities for different values of the RV.

   Example I: $S = \{HH, HT, TH, TT\}$ in tossing of a coin twice with X = number of heads

Value (x)	0	1	2
$P(X = x)$	1/4	1/2	1/4

   Example II: $S = \{NNN, NND, NDN, NDD, DNN, DND, DDN, DDD\}$ in testing electronic components with Y = number of defective components.

Value (x)	0	1	2	3
$P(X = x)$	1/8	3/8	3/8	1/8

Random Variables (4)

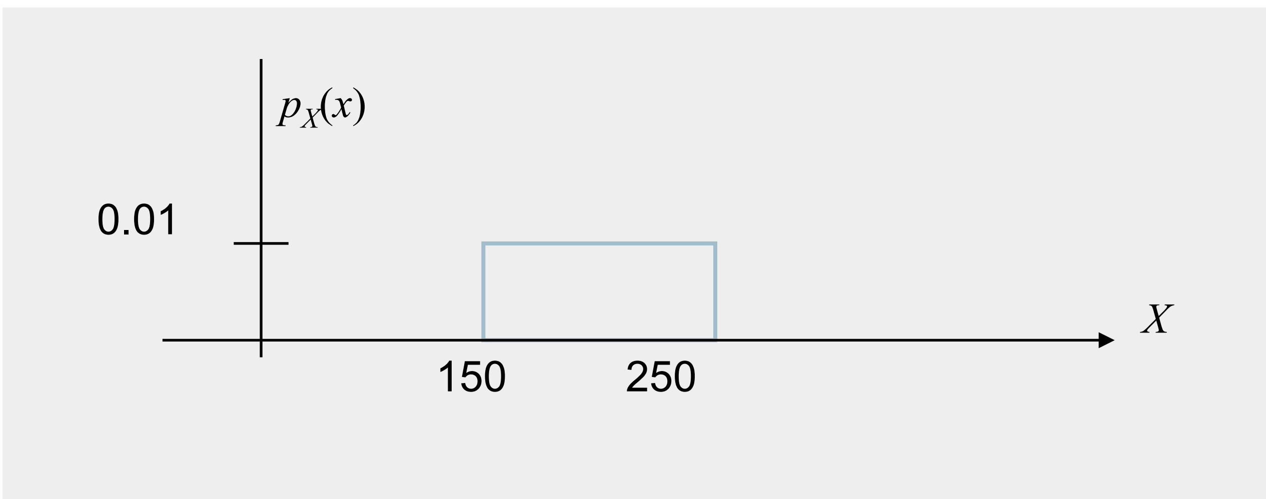
7. Properties:

$a.$	$p_X(x) \geq 0$	always positive
$b.$	$\sum_x p_X(x) = 1$	adds to 1
$c.$	$P(X = x) = p_X(x)$	probability

Random Variables (5)

8. Probability density function of a continuous RV: is represented as a continuous function of *X*.

   Example III: Distance traveled by a car in 5 hours has an uniform distribution between 150 and 250 km.



Random Variables (6)

9. Properties of PDF:

$$a. \quad p_X(x) \geq 0 \qquad \text{always positive}$$

$$b. \quad \int_{-\infty}^{\infty} p_X(x)dx = 1 \qquad \text{integrates to 1}$$

$$c. \quad P(a < X < b) = \int_a^b p_X(x)dx \qquad \text{probability}$$

Random Variables (7)

10. Distribution function: is defined as $F_X(x) = P(X \le x)$

which gives

$$F_X(x) = \sum_{x=-\infty}^{x} p_X(x) \qquad \text{for discrete RV}$$

$$F_X(x) = \int_{-\infty}^{x} p_X(x)dx \qquad \text{for continuous RV}$$

11. Moments:

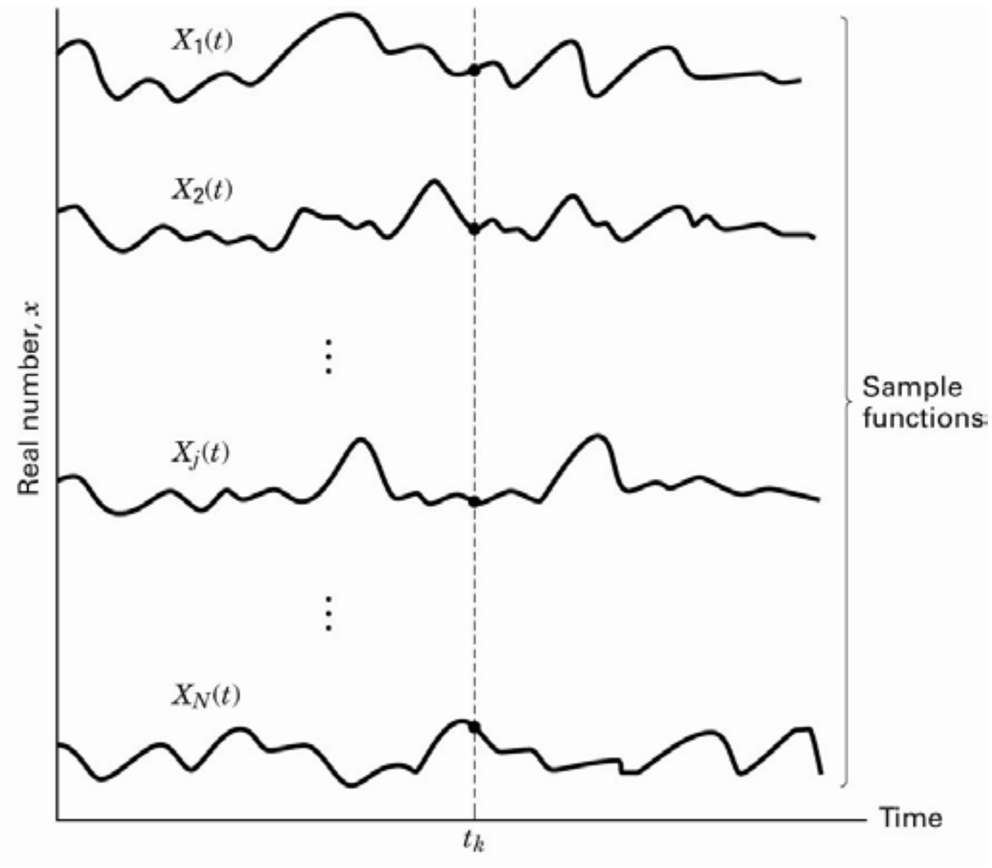
$$E\{X^n\} = \sum_{x=-\infty}^{\infty} x^n p_X(x) \qquad \text{for discrete RV}$$

$$= \int_{-\infty}^{\infty} x^n p_X(x)dx \qquad \text{for continuous RV}$$

12. Mean is defined as $m_X = E\{X\}$. Variance is defined as var$\{X\}$ = $E\{X\}^2 - (m_X)^2$.

Random Processes (1)

1. The outcome of a random process is a time varying function. Examples of ransom processes are: temperature of a room; output of an amplifier; or luminance of a bulb.



Random Processes (2)

2. A random process can also be thought of as a collection of RV's for specified time instants. For example, $X(t_k)$, measured at $t = t_k$ is a RV.
3. Random processes are often specified by their mean and autocorrelation.
4. Mean is defined as

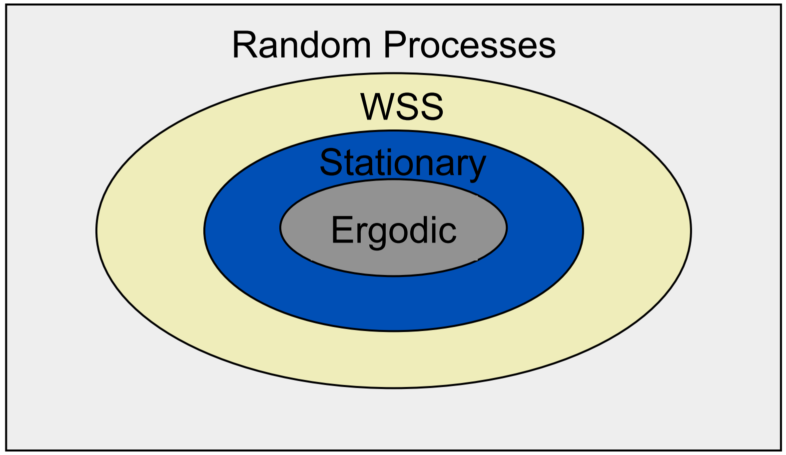
$$E\{X(t_k)\} = \sum_{x=-\infty}^{x} X(t_k) p_{X_k}(x) \qquad \text{for discrete - time random process}$$

$$E\{X(t_k)\} = \int_{-\infty}^{x} X(t_k) p_{X_k}(x) dx \qquad \text{for continuous - time random process}$$

5. Autocorrelation is defined as

$$R_X(t_1, t_2) = E\{X(t_1) X(t_2)\}$$

Classification of Random Processes (1)



1. Wide Sense Stationary (WSS) Process: A random process is said to be WSS if its mean and autocorrelation is not affected with a shift in the time origin

$$E\{x(t)\} = m_X = \text{constant} \quad \text{and} \quad R_X(t_1, t_2) = R_X(t_1 - t_2)$$

2. Strict Sense Stationary (SSS) Process: A random process is said to be SSS if none of its statistics change with a shift in the time origin

$$p_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_k; t_1, t_2, \ldots, t_k) = p_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_k; t_1 + T, t_2 + T, \ldots, t_k + T)$$

3. Ergodic Process: Time averages equal the statistical averages.

Classification of Random Processes (2)

4. For WSS processes, the autocorrelation can be expressed as a function of single variable

$$R_X(t_1,t_2) = R_X(t_1 - t_2) = R_X(\tau)$$

5. Autocorrelation satisfies the following properties

1.	$R_x(\tau) = R_x(-\tau)$	Even function w. r. t. τ
2.	$R_x(\tau) \le R_x(0)$	Maximum occurs at $\tau = 0$
3.	$R_x(\tau) \xleftrightarrow{FT} G_x(f)$	Fourier transform pairs
4.	$R_x(0) = E\left\{X^2(t)\right\}$	Correlation

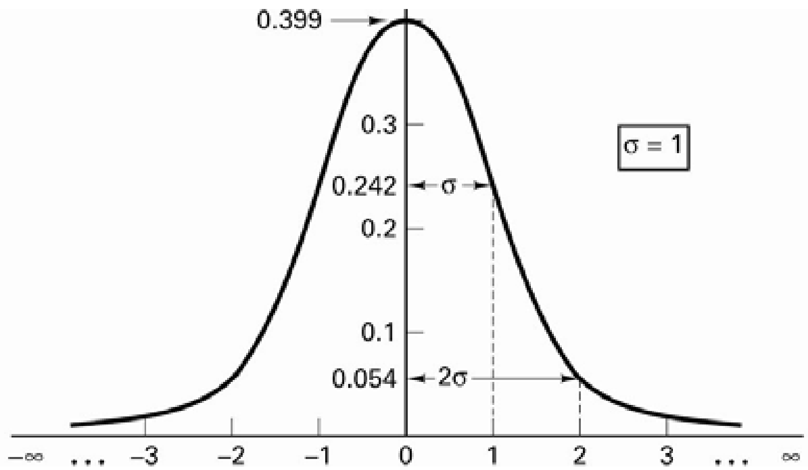
6. Fourier transform of autocorrelation is referred to as the power spectral density (PSD)

1.	$G_x(f) \ge 0$	Always real valued
2.	$G_x(f) = G_x(-f)$	Even function
3.	$R_x(\tau) \xleftrightarrow{FT} G_x(f)$	Fourier transform pairs
4.	$P_X = \int_{-\infty}^{\infty} G_x(f)df$	Variance

Additive Gaussian Noise

1. Noise refers to unwanted interference that tends to obscure the information bearing signal
2. Noise can be classified into two categories:
   a) Man-made Noise introduced by switching transients and simultaneous presence of neighboring signals
   b) Natural Noise produced by the atmosphere, galactic sources, and heating up of electrical components. The latter is referred to as the thermal noise.
3. Thermal noise is difficult to be eliminated and often modeled by the Gaussian probability density function

$$p_N(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2\right]$$



which has a mean $\mu_n = 0$ and var(n) = σ^2.

Additive White Gaussian Noise

4. Additive Gaussian Noise: refers to the following model for introduction of noise in the signal

$$z = a + n \qquad \text{random variable}$$
$$z(t) = A + n(t) \qquad \text{random process}$$

5. Given that the noise *n* is a Gaussian RV and *a* is the DC component, which is constant, the pdf of *z* is given by

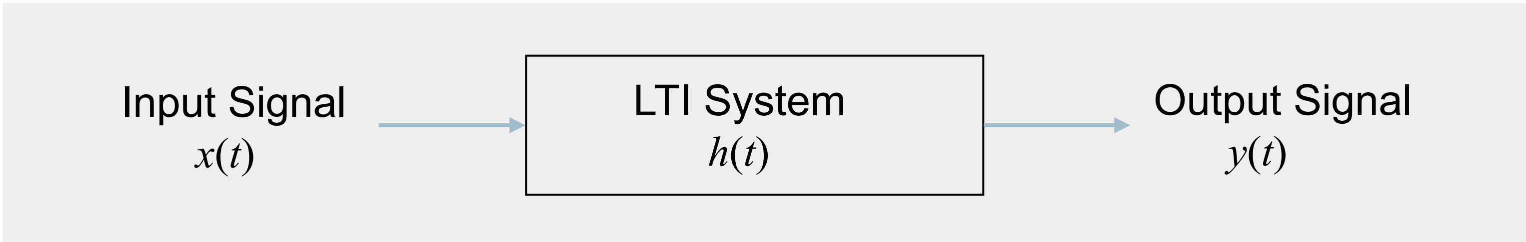
$$p_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a}{\sigma}\right)^2\right]$$

   which has a mean $m_n = a$ and $\text{var}(n) = \sigma^2$.

6. Additive White Gaussian Noise (AWGN): adds an additional constraint on the power spectral density

$$R_n(\tau) = \frac{N_0}{2}\delta(\tau) \xleftrightarrow{FT} G_n(f) = \frac{N_0}{2}$$

Signal Processing with Linear Systems



1. For deterministic signals, the output of the LTI system is given by
   a) Convolution integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\alpha)h(t-\alpha)d\alpha$$

   b) Transfer function:

$$y(t) = \Im^{-1}\left[X(f)H(f)\right]$$

where $X(f)$ and $H(f)$ are Fourier transforms of $x(t)$ and $h(t)$.

Signal Processing with Linear Systems (2)

Input Signal $x(t)$ → LTI System $h(t)$ → Output Signal $y(t)$

2. For WSS random processes, statistics of the output of the LTI system can only be evaluated using the following formula.

$$\text{Mean}: \quad \mu_y = \mu_x \int_{-\infty}^{\infty} h(t)\,dt$$

$$\text{Autocorrelation}: \quad R_y(\tau) = R_x(\tau) * h(\tau) * h(-\tau)$$

$$\text{PSD}: \quad S_y(f) = S_x(f)\,|H(f)|^2$$

Distortionless Transmission

Input Signal $x(t)$ $\rightarrow$ Communication System $h(t)$ $\rightarrow$ Output Signal $y(t)$

1. For distortionless transmission, the signal can only undergo
   a) Amplification or attenuation by a constant factor of K
   b) Time delay of t_0

   In other words, there is no change in the shape of the signal
2. For distortionless transmission, the received signal must be given by

$$y(t) = Kx(t - t_0)$$

3. Based on the above model, the transfer function of the overall communication system is given by

$$H(f) = Ke^{-j2\pi f t_0}$$

with impulse response

$$h(t) = K\delta(t - t_0).$$

Ideal Filters

Lowpass Filter:

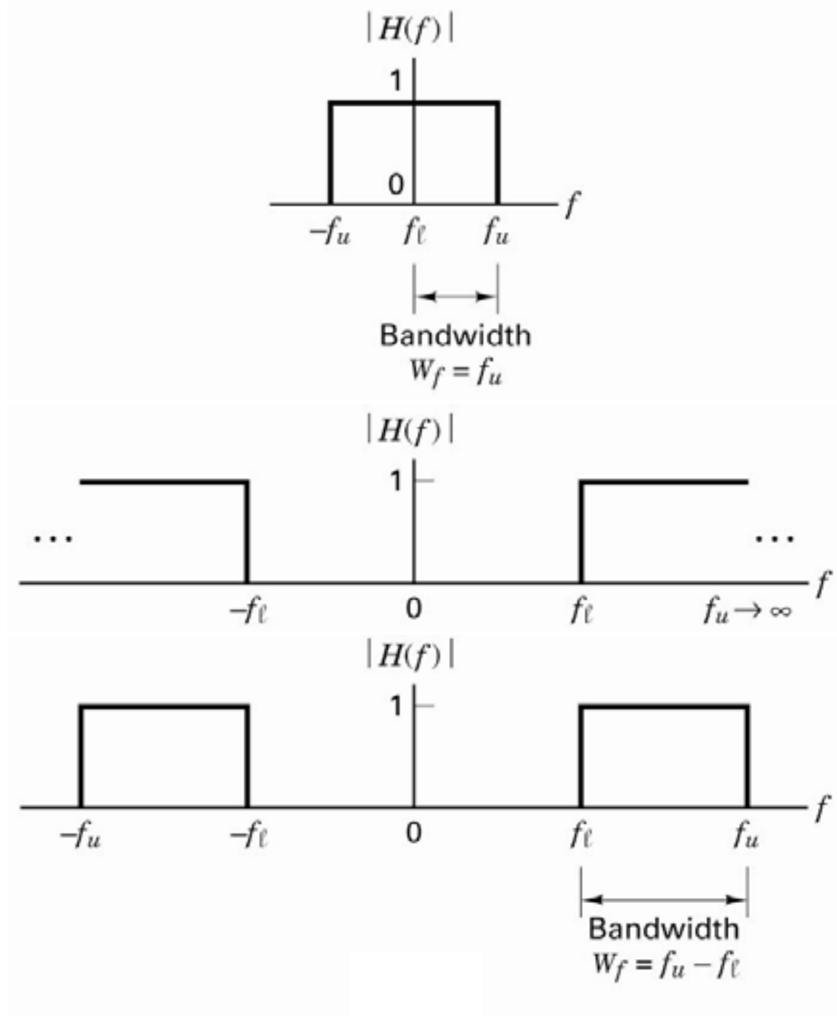
$$H(f) = \begin{cases} e^{-j2\pi f t_0} & |f| < f_u \\ 0 & |f| \geq f_u \end{cases}$$

Highpass Filter:

$$H(f) = \begin{cases} 0 & |f| < f_\ell \\ e^{-j2\pi f t_0} & |f| \geq f_\ell \end{cases}$$

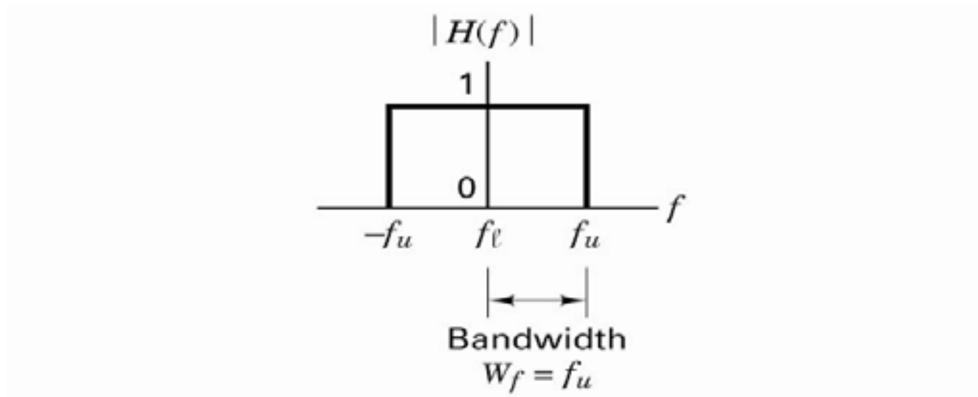
Bandpass Filter:

$$H(f) = \begin{cases} 0 & |f| \leq f_\ell \\ e^{-j2\pi f t_0} & f_\ell < |f| < f_u \\ 0 & |f| \geq f_u \end{cases}$$

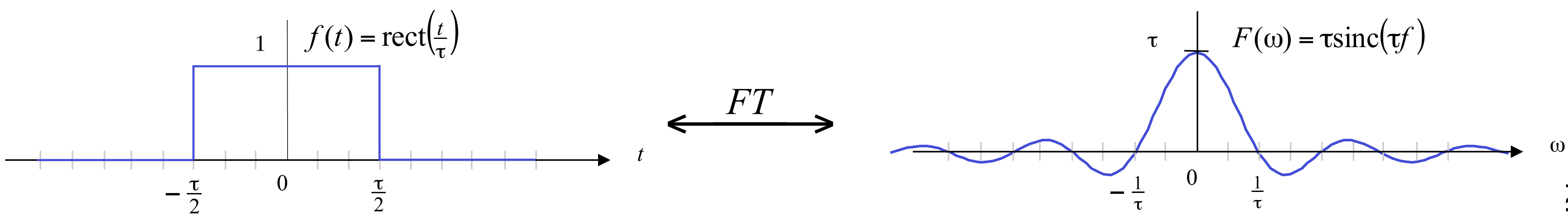


Bandwidth

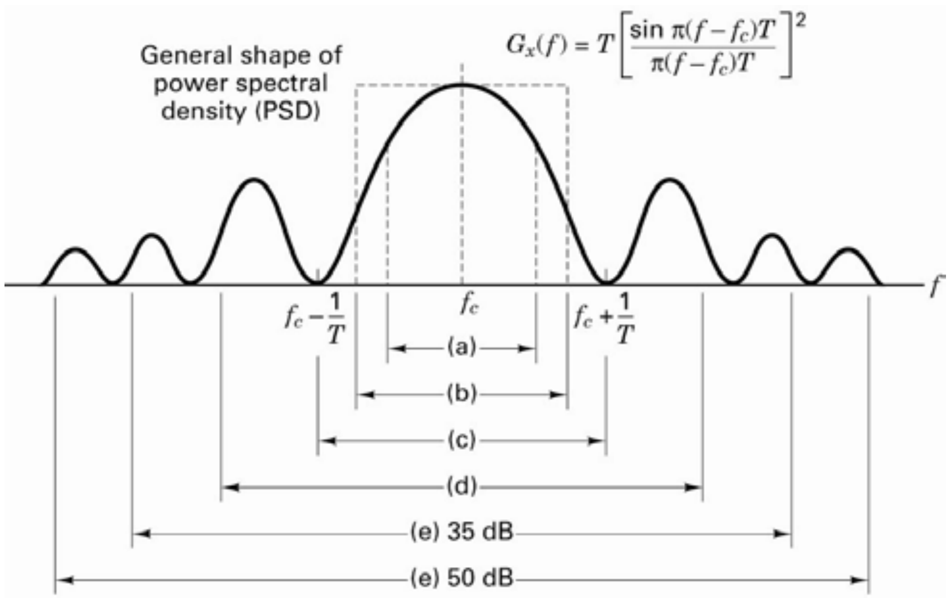
1. For baseband signals, absolute bandwidth is defined as the difference between the maximum and minimum frequency present in a signal.



2. Most time limited signals are not band limited so strictly speaking, their absolute bandwidth approaches infinity



Bandwidth for Bandpass Signals(1)



Alternate definitions of bandwidth include:

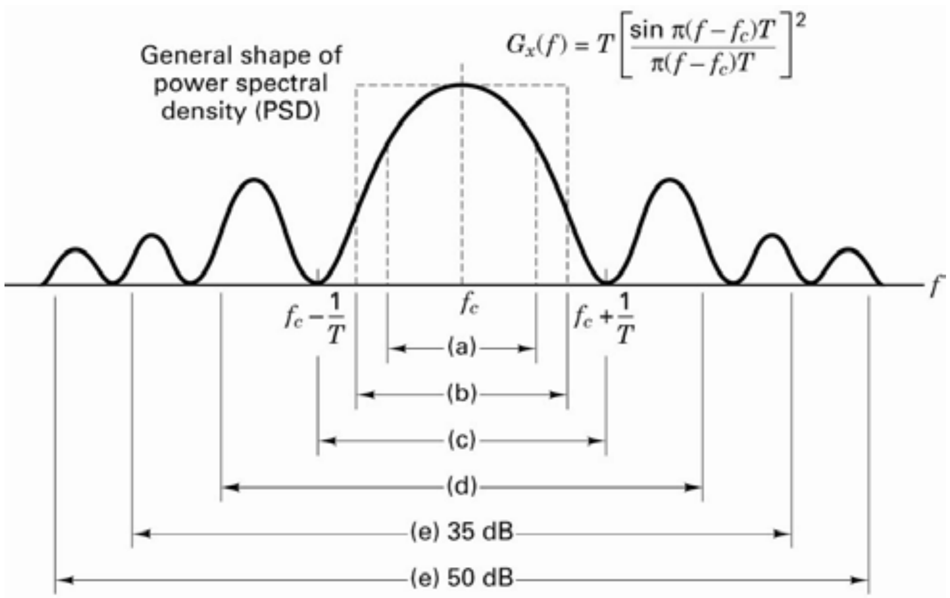
(a) Half-power Bandwidth: Interval between frequencies where PSD drops to 0.707 (3dB) of the peak value.

(b) Noise Equivalent Bandwidth is the ratio of the total signal power (P_x) over all frequencies to the maximum value of PSD $G_x(f_c)$.

(c) Null to Null Bandwidth: is the width of the main spectral lobe.

(d) Fractional Power Containment Bandwidth: is the frequency band centered around f_c containing 99% of the signal power

Bandwidth for Bandpass Signals(2)



Alternate definitions of bandwidth include:

(e) Bounded Power Spectral Density: the width of the band outside which the PSD has dropped to a certain specified level (35dB, 50dB) of the peak value.

(f) Absolute Bandwidth: Band outside which the PSD = 0.