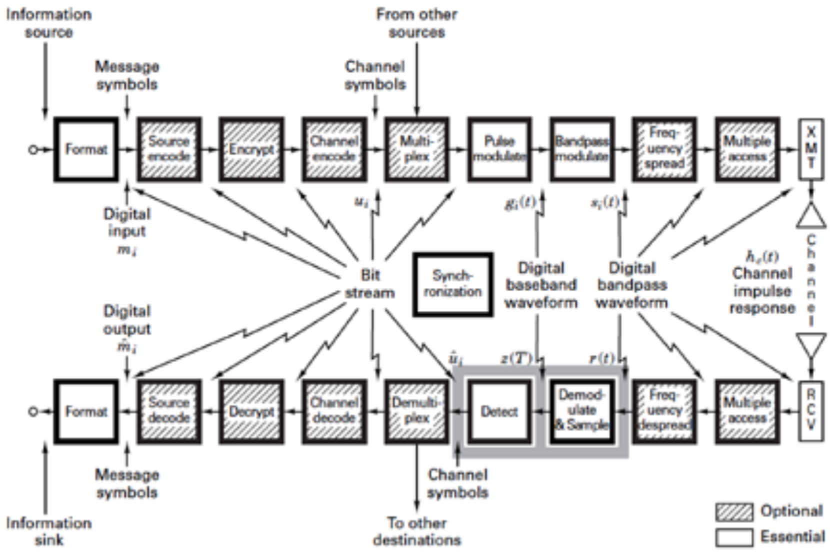
CSE4214 Digital Communications

Chapter 3

Baseband Demodulation/ Detection

Baseband Demodulation/Detection



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CSE4214 Digital Communications

Demodulation and Detection

Why Baseband Demodulation/Detection ?

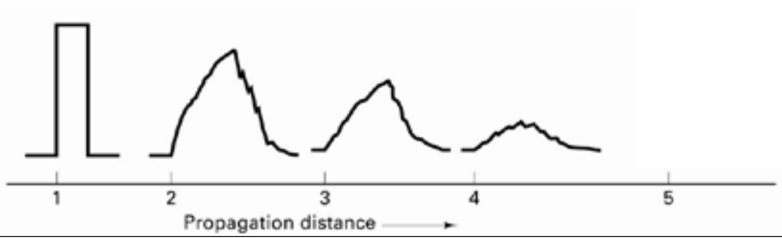
- Received pulses are distorted because of the following factors:

1. Intersymbol Interference causes smearing of the transmitted pulses.
2. Addition of channel noise degrades the transmitted pulses.
3. Transmission channel causes further smearing of the transmitted pulses.

- Demodulation (Detection) is the process of determining the transmitted bits from the distorted waveform.

Transmitted waveform

Received waveforms as a function of distance



Models for Transmitted and Received Signals

- For binary transmission, the transmitted signal over a symbol interval (0, T) is modeled by

$$s_i(t) = \begin{cases} s_1(t) & 0 \le t \le T \quad \text{for bit 1} \\ s_2(t) & 0 \le t \le T \quad \text{for bit 0} \end{cases}$$

- The received signal is degraded by: (i) noise $n(t)$ and (ii) impulse response of the channel

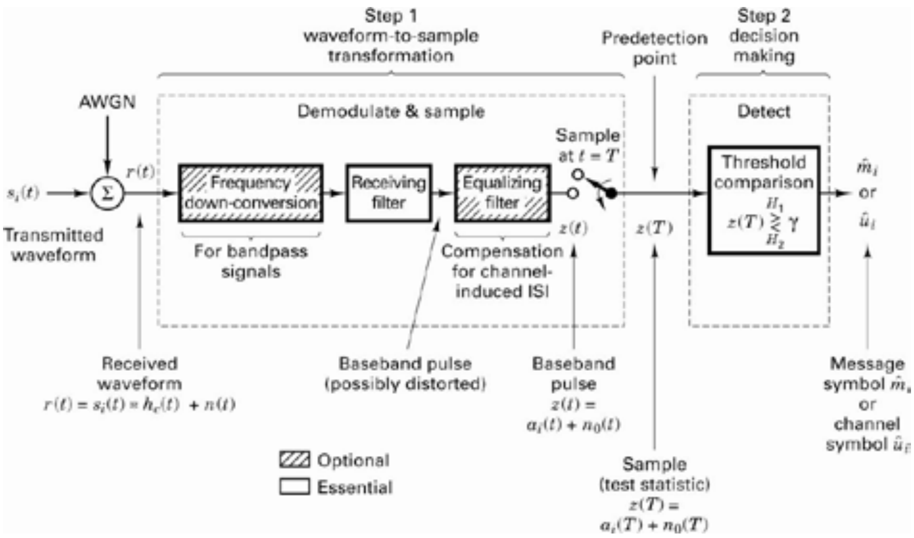
$$r(t) = \underbrace{s_i(t)}_{\substack{\text{transmitted} \\ \text{signal}}} \otimes \underbrace{h_c(t)}_{\substack{\text{channel} \\ \text{impulse} \\ \text{response}}} + \underbrace{n(t)}_{\text{AWGN}}$$

  for $i = 1, \ldots, M$.
- Given $r(t)$, the goal of demodulation is to detect if bit 1 or bit 0 was transmitted.
- In our derivations, we will first use a simplified model for received signal

$$r(t) = \underbrace{s_i(t)}_{\substack{\text{transmitted} \\ \text{signal}}} + \underbrace{n(t)}_{\text{AWGN}}$$

- Later, we will see that degradation due to the impulse response of the channel is eliminated by equalization.

Basic Steps in Demodulation



A Vector View of CT Waveforms (1)

1. Orthonormal Waveforms: Two waveforms $\psi_1(t)$ and $\psi_2(t)$ are orthonormal if they satisfy the following two conditions

Orthogonality Condition: $\int_0^T \psi_1(t)\psi_2(t)\,dt = 0 \quad (0 \le t \le T)$

Unit Magnitude Condition: $\int_0^T \psi_1(t)\psi_1(t)\,dt = K_1 = 1 \quad (0 \le t \le T)$

$\int_0^T \psi_2(t)\psi_2(t)\,dt = K_2 = 1 \quad (0 \le t \le T)$

Normalized to have unit energy

2. Two arbitrary signals s1(t) and s2(t) can be represented by linear combinations of two orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$, i.e.

$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t)$$
$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t)$$
$$\text{where } s_{ij} = \int_0^T s_i(t)\phi_j(t)\,dt, \quad i,j \in \{1,2\}$$

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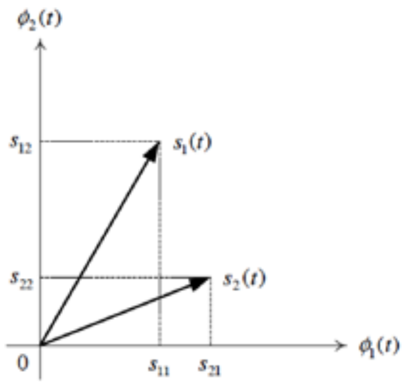
Geometric Representation of Signals

$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t)$$
$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t)$$
$$\text{where } s_{ij} = \int_0^T s_i(t)\phi_j(t)\,dt, \quad i,j \in \{1,2\}$$

$\int_0^T s_i(t)\phi_j(t)dt$ is the projection of s_i on to



A Vector View of CT Waveforms (2)

1. *N*-dimensional basis functions: consists of a set $\{\psi_i(t)\}$, $(1 \le i \le N)$, of orthonormal (or orthogonal) waveforms.

Orthogonality Condition: $$\int_0^T \psi_j(t)\psi_k(t)\,dt = K_j\delta_{jjk} \quad (0 \le t \le T, j,k = 1,\ldots N)$$

where the operator: $$\delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{otherwise} \end{cases}$$

2. Given a basis function, any waveform in the represented as a linear combination of the basis functions.

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A Vector View of CT Waveforms (3)

3. Given a basis function, any waveform in the represented as a linear combination of the basis functions

$$\begin{aligned} s_1(t) &= a_{11}\psi_1(t) + a_{12}\psi_2(t) + \cdots + a_{1N}\psi_N(t) \\ s_2(t) &= a_{21}\psi_1(t) + a_{22}\psi_2(t) + \cdots + a_{2N}\psi_N(t) \\ &\vdots \\ s_M(t) &= a_{M1}\psi_1(t) + a_{M2}\psi_2(t) + \cdots + a_{MN}\psi_N(t) \end{aligned}$$

or, in a more compact form,

$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \quad i = 1,\ldots,M \quad N \le M$$

4. The coefficient a_{ij} are calculated as follows

$$a_{ij} = \frac{1}{K_j}\int_0^T s_i(t)\psi_j(t)\,dt \quad i = 1,\ldots,M \quad j = 1,\ldots,N \quad (0 \le t \le T)$$

where K_j is the energy present in the basis signal.

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Activity 1

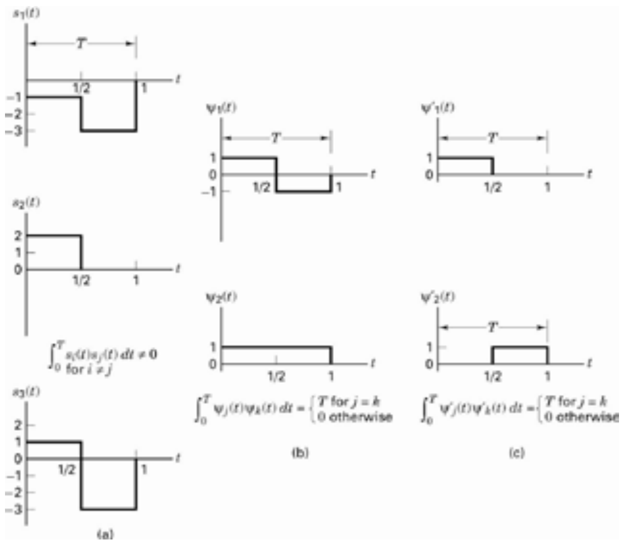
(a) Demonstrate that signals $s_i(t)$, $i = 1,2,3$, are not orthogonal.

(b) Demonstrate that $\psi_i(t)$, $i = 1,2$, are orthogonal.

(c) Express $s_i(t)$, $i = 1,2,3$, as a linear combination of the basis functions $\psi_i(t)$, $i = 1,2$.

(d) Demonstrate that $\psi'_i(t)$, $i = 1,2$, are orthogonal.

(e) Express $s_i(t)$, $i = 1,2,3$, as a linear combination of the basis functions $\psi'_i(t)$, $i = 1,2$.



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Activity 2

- Show that the energy in a signal $s_i(t)$ is given by

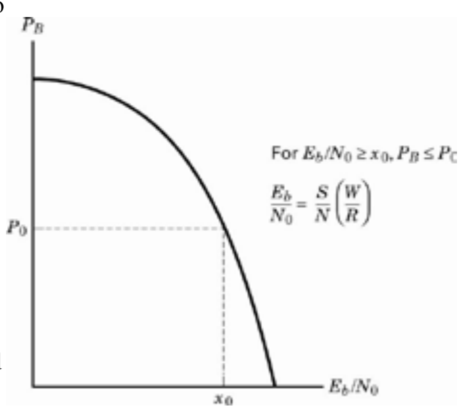
$$E_i = \int_0^T s_i^2(t)dt = \sum_{j=1}^{N} a_{ij}^2 K_j .$$

SNR used in Digital Communications

1. In digital communications, SNR is defined as the ratio of the energy (E_b) present in the signal representing a bit to the power spectral density (N_0) of noise.
2. In terms of signal power S and the duration T of bit, the bit energy is given by $E_b = S \times T$.
3. In terms of noise power N and bandwidth W, the PSD of noise is given by $N_0 = N / W$.
4. SNR is therefore given by

$$SNR = \frac{E_b}{N_0} = \frac{S \times T}{N / W} = \frac{S / R_b}{N / W} = \frac{S}{N}\frac{W}{R}$$

   where R_b is the rate of transmission in bits transmitted per second (bps).
5. Bit-error probability is the probability of error in a transmitted bit.
6. ROC curves are plots of Bit-error probability versus SNR.



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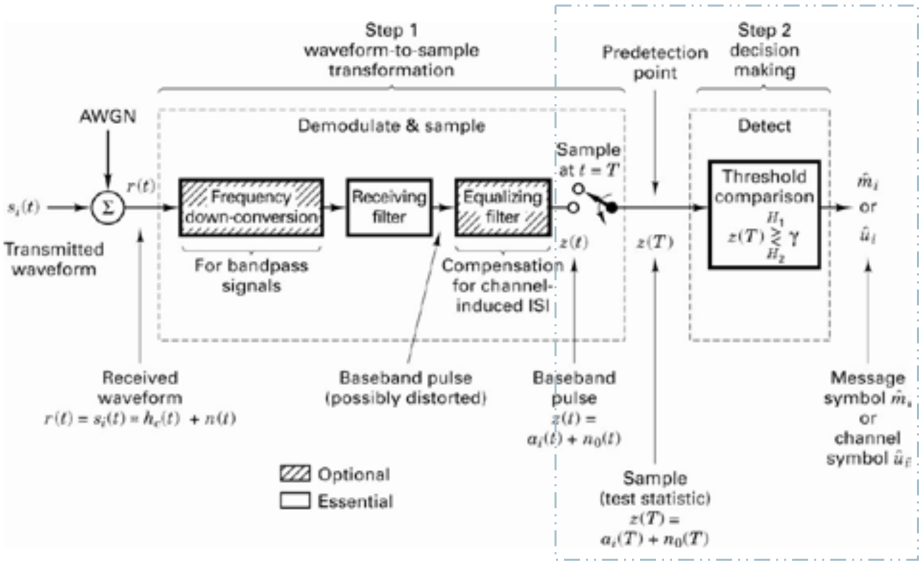
Detection of binary signal in AWGN



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Maximum Likelihood Detector (1)

1. The sampled received signal is given by

$$z(T) = r(t)\big|_{t=T} = \underbrace{a_i(T)}_{\text{Level of signal}} + \underbrace{n_0(T)}_{\text{Gaussian RV}}$$

where $a_i(T)$ represents th... For bit 1, $a_i(T) = a_1$ and for bit 0, $a_i(T) = a_2$.

2. The pdf of n_0 is Gaussian with zero mean

$$p(n_0) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n_0}{\sigma_0}\right)^2\right]$$

3. The conditional pdf of z gi... d is given by

$$p(z|s_1) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a_1}{\sigma_0}\right)^2\right]$$

$$p(z|s_2) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2\right]$$

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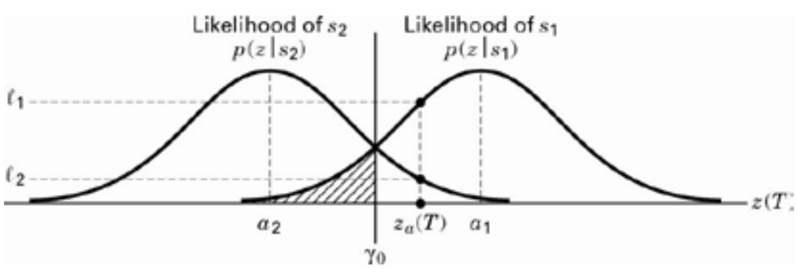
Maximum Likelihood Detector (2)

4. The conditional pdf of z given that bit 1 or bit 0 was transmitted are referred to as maximum likelihoods.

$$p(z|s_1) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a_1}{\sigma_0}\right)^2\right] \quad \text{maximum likelihood of } s_1$$

$$p(z|s_2) = \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2\right] \quad \text{maximum likelihood of } s_2$$

5. The maximum likelihoods have the following distributions



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Maximum Likelihood Detector (1)

6. Maximum Likelihood Ratio Test: is given by

$$\frac{p(z|s_1)}{p(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

where $P(s_1)$ and $P(s_2)$ are the priori probabilities that $s_1(t)$ and $s_2(t)$, respectively, are transmitted.

H_1 and H_2 are two possible hypotheses. H_1 states that signal $s_1(t)$ was transmitted and hypothesis H_2 states that signal $s_2(t)$ was transmitted.

6. For $P(s_1) = P(s_2)$, the maximum likelihood ratio test reduces to

$$z(T) \underset{H_2}{\overset{H_1}{\gtrless}} \frac{a_1 + a_2}{2} = \gamma_0$$

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Activity 3

- Show that the probability of bit error for maximum likelihood ratio test is given by

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$$

$$\text{where } Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{u^2}{2}\right) dx$$

$$\text{or } Q(x) \approx \frac{1}{x\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \quad \text{for } x > 3$$

- Values of Q(x) are listed in Table B.1 in Appendix B page 1046.

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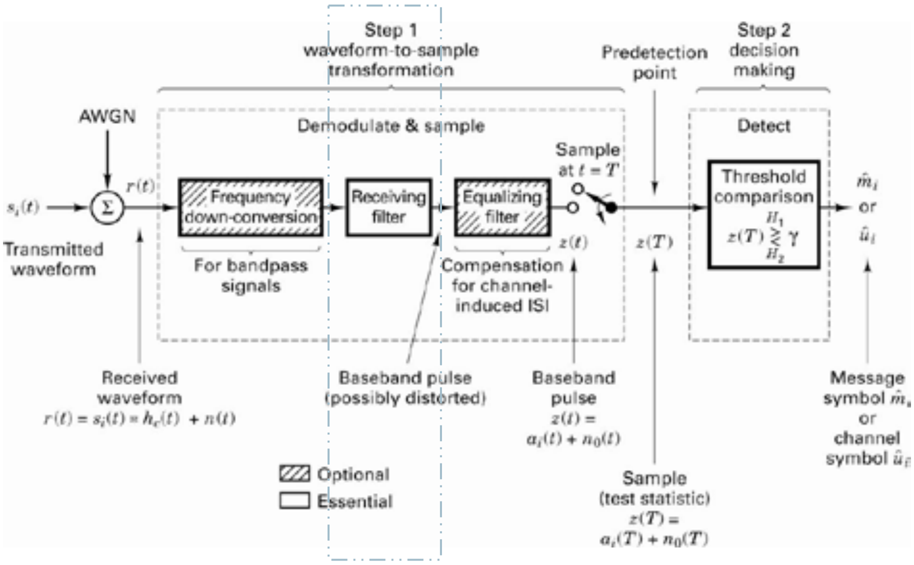
Matched Filtering (1)



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Matched Filtering (2)

$$r(t) = s_i(t) + n(t) \longrightarrow \boxed{h(t) \xleftrightarrow{\text{CTFT}} H(f)} \longrightarrow Z(T) = a_i(T) + n_0(T)$$

Design the Receiving filter $h(t)$

1. Design a filter that maximizes the SNR at time ($t = T$) of the sampled signal

$$z(T) = r(t)\big|_{t=T} = \underbrace{a_i(T)}_{\text{Level of signal}} + \underbrace{n_0(T)}_{\text{Gaussian RV}}$$

2. The instantaneous signal power to noise power is given by

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2}$$

where $a_i(t)$ is the filtered signal and σ_0^2 is the variance of the output noise

3. The information bearing component is given by

$$a_i(T) = \int_{-\infty}^{\infty} H(f) S_i(f) e^{j2\pi ft} dt$$

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Matched Filtering (3)

4. Given that input noise $n(t)$ is AWGN with $S_n(f) = N_0/2$, the PSD of the output noise is given by

$$S_{n0}(f) = |H(f)|^2 S_n(f) = \frac{N_0}{2} S_n(f)$$

5. The output noise power is given by

$$\sigma_0^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$$

6. The SNR is given by

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left|\int_{-\infty}^{\infty} H(f) S(f) e^{j2\pi ft} df\right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

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Matched Filtering (4)

Schwartz Inequality:

$$\left|\int_{-\infty}^{\infty} f_1(x) f_2(x) dx\right|^2 \le \int_{-\infty}^{\infty} |f_1(x)|^2 dx \int_{-\infty}^{\infty} |f_2(x)|^2 dx$$

with the equality valid if $f_1(x) = kf_2^*(x)$.

7. Applying the Schwatz inequality to the SNR gives

$$\left(\frac{S}{N}\right)_T \le \frac{\int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} \left|S(f) e^{j2\pi ft}\right|^2 df}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

or,

$$\left(\frac{S}{N}\right)_T \le \frac{2}{N_0} \int_{-\infty}^{\infty} |S(f)|^2 df$$

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Matched Filtering (5)

8. The maximum SNR is given by

$$\max\left(\frac{S}{N}\right)_T = \frac{2}{N_0}\int_{-\infty}^{\infty}|S(f)|^2\,df = \frac{2E}{N_0}$$

which is possible if

$$H(f) = S^*(f)e^{-j2\pi ft} \quad \text{or} \quad h(t) = s(T-t).$$

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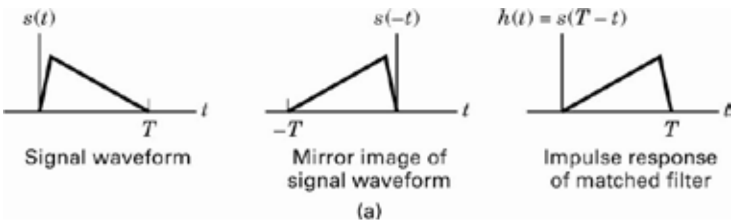
Activity 4

- Given that the transmitted signal is shown in the following figure, determine the impulse response of the matched filter.



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Matched Filtering (6)

Correlator Implementation of matched filter:

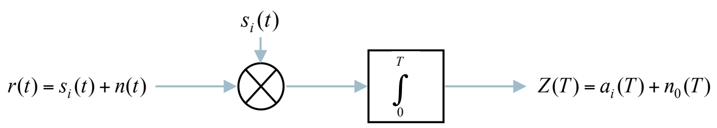
The output of the matched filter is given by

$$z(t) = \int_0^t r(\tau)h(t-\tau)d\tau$$

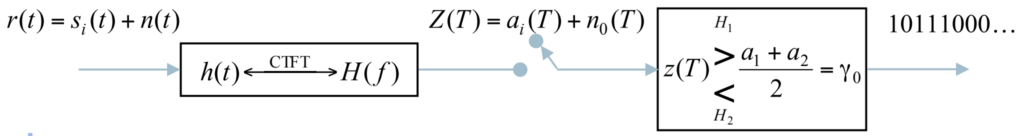
The output at $t = T$ is given by

$$z(T) = \int_0^T r(\tau)h(T-\tau)d\tau = \int_0^T r(\tau)s_i(\tau)d\tau$$

which leads to the following correlator implementation for matched filter



Detection of binary signal: Review



Matched Filter:

Impulse response: $h(t) = s_i(T-t)$

Maximum SNR = $(a_1)^2/\sigma^2_0 = 2E_s/N_0$

ML Detector:

Probability of Error:

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$$

- The overall goal of the receiver should be to minimize the probability of bit error P_B..
- In other words, we are interested in maximizing $(a_1 - a_2)^2/\sigma^2_0$.
- The filter is designed such that it is matched to the difference of $[s_1(t) - s_2(t)]$.
- Maximum SNR of the matched filter = $(a_1 - a_2)^2/\sigma^2_0 = 2(E_{s1} - E_{s2})/N_0 = 2E_d/N_0$.
- Probability of Error:

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right) = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

Activity 5

By defining the cross-correlation coefficient as

$$\rho = \frac{1}{E_b}\int_{-\infty}^{\infty} s_1(t)s_2(t)dt \quad \text{with} \quad E_b = \int_{-\infty}^{\infty} s_1^2(t)dt = \int_{-\infty}^{\infty} s_2^2(t)dt$$

show that the probability of bit error using a filter matched to $[s_1(t) - s_2(t)]$ is given by

$$P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

Using the above relationship, show that the probability of bit error is given by

1. $P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$ for antipodal signals : $s_1(t) = -s_2(t)$
2. $P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$ for orthogonal signals $s_1(t) \perp s_2(t)$.

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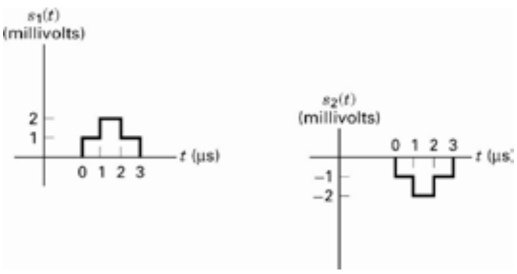
Activity 6

Determine the probability of bit error for a binary communication system, which receives equally likely signals $s_1(t)$ and $s_2(t)$ shown in the following diagram



Assume that the receiving filter is a matched filter and the power spectral density of AWGN is $N_0 = 10^{-12}$ Watt/Hz.

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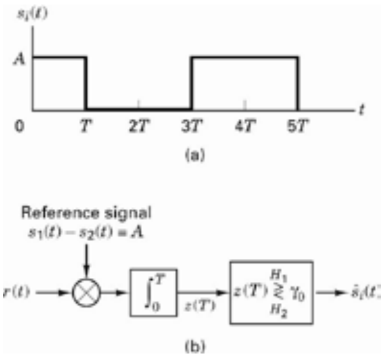
Error Probability Performance of Binary Signal

1. Unipolar signaling

In Unipolar signaling, the signal selection to represent bits 1 and 0 is as follows:

$$s_1(t) = A, \quad (0 \le t \le T), \quad \text{for bit 1}$$
$$s_2(t) = 0, \quad (0 \le t \le T), \quad \text{for bit 0.}$$

The receiver is shown by the following diagram.



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Unipolar Signaling

- Unipolar signal forms an orthogonal signal set.
- When $s_1(t)$ plus AWGN being received, the expected value of $z(T)$, given that $s_1(t)$ was sent, is

$$a_1(T) = E\left\{z(T)|s_1(t)\right\} = E\left\{\int_0^T (A^2 + An(t))\,dt\right\} = A^2T, \text{ where } E\left\{n(t)\right\} = 0$$

- When $s_2(t)$ plus AWGN being received, $a_2(T)$=0.
- The optimum decision threshold is:

$$\gamma_0 = \frac{a_1 + a_2}{2} = \frac{1}{2}A^2T$$

- The bit-error performance is:

$$P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right), \text{ where } E_b = \frac{A^2T}{2}$$

$$E_d = \int_0^T [s_1(t) - s_2(t)]^2\,dt = A^2T$$

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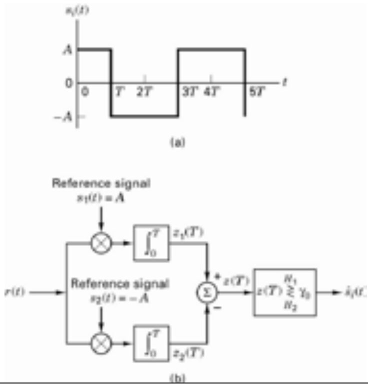
Bipolar Signaling

In bipolar signaling, the signal selection to represent bits 1 and 0 is as follows:

$$\begin{aligned} s_1(t) &= A, \quad (0 \le t \le T), \quad \text{for bit 1} \\ s_2(t) &= -A, \quad (0 \le t \le T), \quad \text{for bit 0.} \end{aligned}$$

The receiver is shown by the following diagram.



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Bipolar Signaling

- Bipolar signal is a set of antipodal signal, e.g. $s_1(t)$=-$s_2(t)$.
- The optimum decision threshold is:

$$\gamma_0 = \frac{a_1 + a_2}{2} = 0$$

- The bit-error performance is:

$$P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right), \text{ where } E_b = A^2T$$

$$E_d = \int_0^T \left[s_1(t) - s_2(t)\right]^2 dt = (2A)^2 T$$

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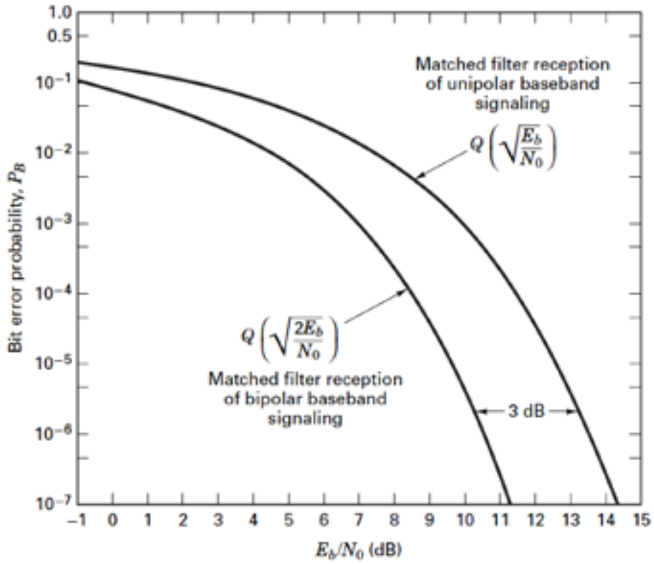
Unipolar vs. Bipolar



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