CSE4214 Digital Communications

Chapter 3

Class Activities

Activity 1

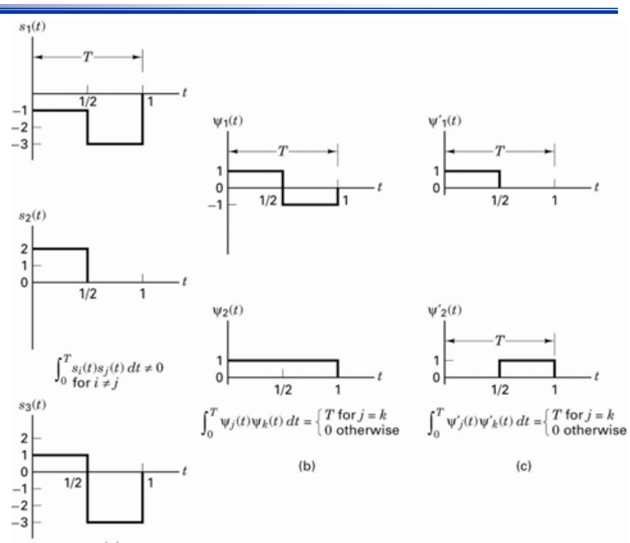
(a) Demonstrate that signals $s_i(t)$, i = 1,2,3, are not orthogonal.

(b) Demonstrate that $\psi_i(t)$, i = 1,2, are orthogonal.

(c) Express $s_i(t)$, i = 1,2,3, as a linear combination of the basis functions $\psi_i(t)$, i = 1,2.

(d) Demonstrate that $\psi'_i(t)$, i = 1,2, are orthogonal.

(e) Express $s_i(t)$, i = 1,2,3, as a linear combination of the basis functions $\psi'_i(t)$, i = 1,2.



2

2

Activity 1

(a) $s_1(t)$, $s_2(t)$, and $s_3(t)$ are clearly not orthogonal, i.e. the time integrated value over a symbol duration of the product of any two of the three waveforms is not zero, e.g.

$$\int_0^T s_1(t)s_2(t)dt = \int_0^{\frac{T}{2}} s_1(t)s_2(t)dt + \int_{\frac{T}{2}}^T s_1(t)s_2(t)dt$$

$$= \int_0^{\frac{T}{2}} (-1)(2)dt + \int_{\frac{T}{2}}^T (-3)(0)dt = -T$$

(b) Using the same method, we can verify this signal form an orthogonal set.

$$\int_0^T \psi_1(t)\psi_2(t)dt = \int_0^{\frac{T}{2}} \psi_1(t)\psi_2(t)dt + \int_{\frac{T}{2}}^T \psi_1(t)\psi_2(t)dt$$

$$= \int_0^{\frac{T}{2}} (1)(1)dt + \int_{\frac{T}{2}}^T (-1)(1)dt = 0$$

Activity 1

(c) We know that $a_{ij} = \frac{1}{K_j}\int_0^T s_i(t)\psi_j(t)dt \quad i=1,\ldots,M \quad j=1,\ldots,N \quad (0 \le t \le T)$

$$a_{11} = \frac{1}{T}\left\{\int_0^{\frac{T}{2}} s_1(t)\psi_1(t)dt + \int_{\frac{T}{2}}^T s_1(t)\psi_1(t)dt\right\} = \frac{1}{T}\left\{\int_0^{\frac{T}{2}} (-1)(1)dt + \int_{\frac{T}{2}}^T (-3)(-1)dt\right\} = 1$$

$$a_{12} = \frac{1}{T}\left\{\int_0^{\frac{T}{2}} s_1(t)\psi_2(t)dt + \int_{\frac{T}{2}}^T s_1(t)\psi_2(t)dt\right\} = \frac{1}{T}\left\{\int_0^{\frac{T}{2}} (-1)(1)dt + \int_{\frac{T}{2}}^T (-3)(1)dt\right\} = -2$$

$$a_{21} = 1, \quad a_{22} = 1, \quad a_{31} = 2, \quad a_{32} = -1$$

$$s_1(t) = \psi_1(t) - 2\psi_2(t)$$
$$s_2(t) = \psi_1(t) + \psi_2(t)$$
$$s_3(t) = 2\psi_1(t) - \psi_2(t)$$

Activity 1

(d) $\psi_1^{'}(t), \psi_2^{'}(t)$ are obviously orthogonal.

(e) Similar to (c), we have

$$s_1(t) = -\psi_1^{'}(t) - 3\psi_2^{'}(t)$$
$$s_2(t) = 2\psi_1^{'}(t)$$
$$s_3(t) = \psi_1^{'}(t) - 3\psi_2^{'}(t)$$

Activity 2

- Show that the energy in a signal $s_i(t)$ is given by

$$E_i = \int_0^T s_i^2(t)dt = \sum_{j=1}^{N} a_{ij}^2 K_j.$$

Activity 2

$$E_i = \int_0^T s_i^2(t)dt = \int_0^T \left[\sum_j a_{ij}\psi_j(t) \right]^2 dt$$

$$= \int_0^T \sum_j a_{ij}\psi_j(t) \sum_k a_{ik}\psi_k(t)dt = \sum_j \sum_k a_{ij}a_{ik} \int_0^T \psi_j(t)\psi_k(t)dt$$

$$= \sum_j \sum_k a_{ij}a_{ik}K_j\delta_{jk} = \sum_{j=1}^N a_{ij}^2 K_j \quad i = 1,\ldots,M$$

Activity 3

- Show that the probability of bit error for maximum likelihood ratio test is given by

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0} \right)$$

$$\text{where } Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2} \right) du$$

$$\text{or } Q(x) \approx \frac{1}{x\sqrt{2\pi}} \exp\left(\frac{-x^2}{2} \right) \quad \text{for } x > 3$$

- Values of Q(x) are listed in Table B.1 in Appendix B page 1046.

8

Activity 3

Show that the probability of bit error for maximum likelihood ratio test is given by

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$$

The probability of a bit error is numerically equal to the area under the "tail" of either likelihood function.

$$P_B = \int_{\gamma_0}^{\infty} p(z|s_2)dz = \int_{\gamma_0}^{\infty} \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2\right]dz$$

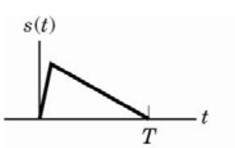
Let $u = \dfrac{z-a_2}{\sigma_0}$, then $\sigma_0 du = dz$,

$$P_B = \int_{\frac{(a_1-a_2)}{2\sigma_0}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) = Q\left(\frac{a_1-a_2}{2\sigma_0}\right)$$

9

Activity 4

- Given that the transmitted signal is shown in the following figure, determine the impulse response of the matched filter.

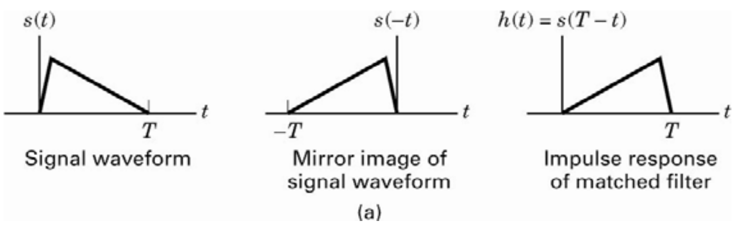


Signal waveform

10

Activity 4

- Given that the transmitted signal is shown in the following figure, determine the impulse response of the matched filter.



11

Activity 5

By defining the cross-correlation coefficient as

$$\rho = \frac{1}{E_b}\int_{-\infty}^{\infty} s_1(t)s_2(t)dt \quad \text{with} \quad E_b = \int_{-\infty}^{\infty} s_1^2(t)dt = \int_{-\infty}^{\infty} s_2^2(t)dt$$

show that the probability of bit error using a filter matched to [$s_1(t) - s_2(t)$] is given by

$$P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

Using the above relationship, show that the probability of bit error is given by

1. $P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$ for antipodal signals: $s_1(t) = -s_2(t)$
2. $P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$ for orthogonal signals $s_1(t) \perp s_2(t)$.

12

Activity 5

$$\rho = \frac{1}{E_b}\int_{-\infty}^{\infty} s_1(t)s_2(t)dt \quad \text{with} \quad E_b = \int_{-\infty}^{\infty} s_1^2(t)dt = \int_{-\infty}^{\infty} s_2^2(t)dt$$

Show that the probability of bit error using a filter matched to $[s_1(t) - s_2(t)]$ is given by

$$P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

We know

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right) = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

$$\because E_d = \int_0^T \left[s_1(t) - s_2(t)\right]^2 dt = \int_0^T s_1^2(t)dt + \int_0^T s_2^2(t)dt - 2\int_0^T s_1(t)s_2(t)dt$$

$$= E_b + E_b - 2\rho E_b = 2E_b(1-\rho)$$

$$\therefore \quad P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b(1-\rho)}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

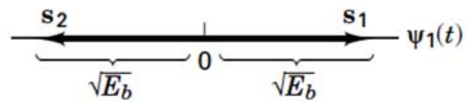
13

Activity 5

$$P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

Using the above relationship, show that the probability of bit error is given by

$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad \text{for antipodal signals: } s_1(t) = -s_2(t)$$



$$\rho = \frac{1}{E_b}\int_{-\infty}^{\infty} s_1(t)s_2(t)dt \quad \text{with} \quad E_b = \int_{-\infty}^{\infty} s_1^2(t)dt = \int_{-\infty}^{\infty} s_2^2(t)dt$$

$$\because \rho = \frac{1}{E_b}\int_0^T s_1(t)s_2(t)dt = \frac{1}{E_b}\int_0^T \left[-s_1^2(t)\right]dt = -1$$

$$\therefore \quad P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

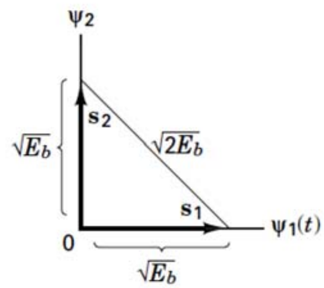
14

Activity 5

$$P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

Using the above relationship, show that the probability of bit error is given by

$P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$ for orthogonal signals $s_1(t) \perp s_2(t)$



$$\rho = \frac{1}{E_b}\int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt \quad \text{with} \quad E_b = \int_{-\infty}^{\infty} s_1^2(t)\,dt = \int_{-\infty}^{\infty} s_2^2(t)\,dt$$

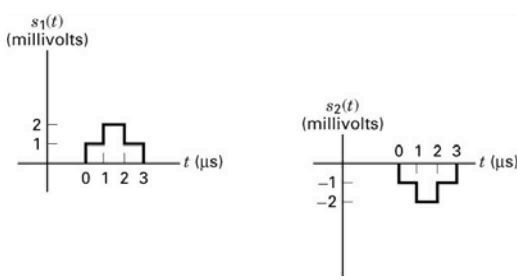
$s_1(t)$ and $s_2(t)$ are orthogonal, $\rho = 0$

$$P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

15

Activity 6

Determine the probability of bit error for a binary communication system, which receives equally likely signals $s_1(t)$ and $s_2(t)$ shown in the following diagram



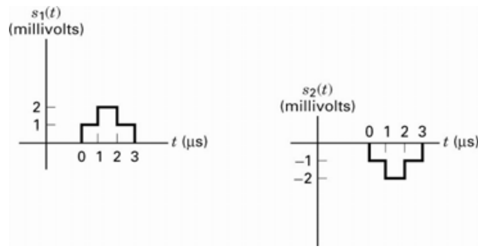
Assume that the receiving filter is a matched filter and the power spectral density of AWGN is $N_0 = 10^{-12}$ Watt/Hz.

16

Activity 6

Determine the probability of bit error for a binary communication system, which receives equally likely signals $s_1(t)$ and $s_2(t)$ shown in the following diagram.

Assume that the receiving filter is a matched filter and the power spectral density of AWGN is $N_0 = 10^{-12}$ Watt/Hz.



$$E_b = \int_0^3 v^2(t)\,dt = (10^{-3}V)^2 \times (10^{-6}s) + (2\times 10^{-3}V)^2 \times (10^{-6}s) + (10^{-3}V)^2 \times (10^{-6}s) = 6\times 10^{-12} \text{ joule}$$

$$P_B = \mathrm{Q}\left(\sqrt{\frac{12\times 10^{-12}}{10^{-12}}}\right) = Q(\sqrt{12}) = Q(3.46) = 3\times 10^{-4}$$

$$\approx \frac{1}{x\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) = 2.9\times 10^{-4}$$

17