PROJECT 1: INTRODUCTION TO SIGNAL AND SPECTRA

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1. PURPOSE

In this lab, you will be introduced to random variables and stochastic processes. Using MATLAB, you will learn how to generate random variables of any known power spectral density (PSD) through a linear, time invariant (LTI) system representation whose impulse response is derived from the PSD. You will also implement a quadrature amplitude modulator and demodulator as a prototype communication system. The lab will also serve as a review of MATLAB.

The learners will complete the following three simulations in MATLAB and submit their solutions along with a soft copy of the code in the form of a report.

2. OBJECTIVES

By the end of this project, you will be able to:

1. Generate random variables of commonly used probability density function (pdf) including uniform and Gaussian pdfs.
2. Derive and plot the autocorrelation function as well as the PSD of a sequence of random variables.
3. Implement a quadrature amplitude modulator and demodulator widely used in many digital radio communications and data communications applications.

3. REFERENCES

1. Bernard Sklar text: Sections 1.5 - 1.7. Pages 20 - 50.

4. INTRODUCTION

As covered in the class, a random variable is a function that associates a real number with each element in the sample space of a random experiment . For example, the sample space giving a detailed description of each possible outcomes when three electronic components are tested may be written as S = NNN, NND, NDN, NDD, DNN, DND, DDN, DDD, where N denotes a nondefective and D denotes a defective component. A natural concern is the number of defective components that occur. If we assign a random variable X that counts the number of defective components, then each event or point in the sample space can be assigned a numerical value of 0, 1, 2, or 3. For the sample point NNN, the value of X is 0 and for DDN, X has a value of 2. Similarly, for the rest of the sample points. The random variable considered in the above example is a discrete random variable because its set of possible values is countable. When a random variable can take on values on the continuous scale, it is called a continuous random variable. Such is the case, for example, when one conducts an investigation measuring the distances that a certain make of automobile travels over a prescribed course in 5 liters of gasoline. Assuming distances to be a random variable measured with any degree of accuracy, say up to two decimal points, then clearly we have an infinite number of possible distances in the sample space. For further discussion on random variables, refer to the handout on random variables available on the course home page.

A (one-dimensional) random process is a scalar function $x(t)$, where t is usually time, for which the future evolution is not determined uniquely by any set of initial dataor at least by any set that is knowable to you and me. In other words, random process is just a fancy phrase that means unpredictable function. Example of a random process is the voltage measured across a resistor in an RLC circuit. The voltage is a function of time hence the notation $v(t)$ used to represent it. Second, each time the voltage is measured you would get a different waveform (referred to as ensemble). Therefore, the sample space of a random process consists of a number of waveforms varying with time instead of a combination of numbers as observed with the random variable. If on the other hand, we fix time $t = t_0$ in the sample space then each ensemble will produce a single number. The combination of these numbers across all ensembles can be treated as a random variable. A particular category of random process is one where the mean of the random variable obtained by fixing time is constant (independent of time) and the autocorrelation between two random variables obtained by fixing time at two different instants is only dependent on the time difference between the two instants. Such a random process is called a wide sense stationary (WSS) process, which has wide applications in digital communication including modeling of noise.

In this lab, you will use MATLAB to generate random variables and random processes through three simulations.

5. GENERATION OF A GAUSSIAN SEQUENCE:

In the first simulation, you are required to generate 1000 pairs of Gaussian random variables (x_1, x_2) that have mean vector

$$m = \mathcal{E}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \tag{1}$$

and covariance matrix

$$C = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \tag{2}$$

where $\mathcal{E}$ is the expectation operator.

Problem 1: *In MATLAB, the function* **`randn`** *generates random variables with Gaussian distribution having mean zero and variance one. Write a function based on* **`randn`** *that will generate the correlated variables (x_1, x_2) defined above.*

Problem 2: *For the random variables generated above, determine the mean of the samples (x_1, x_2), $i = 1, 2, \ldots, 1000$, using the relationships*

$$m_1 = \frac{1}{1000}\sum_{i=1}^{1000} x_{1i} \tag{3}$$

$$m_2 = \frac{1}{1000}\sum_{i=1}^{1000} x_{2i}. \tag{4}$$

Problem 3: *Compare the values obtained from the samples with the theoretical values that were provided to you. Why are the two different? Now, increase the number of samples to 10000. Does this affect your answer in any way? Has the approximation improved?*

Problem 4: *Repeat (b) but now calculate the covariances. The analytical expression for the covariances are*

$$\textit{Variance of } x_1\text{:} \qquad \sigma_1^2 = \frac{1}{1000}\sum_{i=1}^{1000}(x_{1i} - m_1)^2 \tag{5}$$

$$\textit{Variance of } x_2\text{:} \qquad \sigma_2^2 = \frac{1}{1000}\sum_{i=1}^{1000}(x_{2i} - m_2)^2 \tag{6}$$

$$\textit{Covariance of } x_1 \textit{ and } x_2\text{:} \qquad c_{12} = \frac{1}{1000}\sum_{i=1}^{1000}(x_{1i} - m_1)(x_{2i} - m_2). \tag{7}$$

6. FILTERING OF STOCHASTIC SIGNALS:

Problem 5: *Modify the code of simulation 1 to generate an independent identically distributed (i.i.d.) sequence $\{x_n\}$ of $N = 1000$ uniformly distributed random variables in the interval [-0.5, 0.5]. Unlike simulation 1, you are now dealing with a single random variable with uniform distribution in the interval [-0.5, 0.5]. Calculate the mean and variance of the generated sequence and compare with the theoretical values obtained directly from the distribution.*

Problem 6: *The sequence $\{x_n\}$ is passed through a causal and stable linear filter with impulse response*

$$h[n] = \begin{cases} (0.95)^n, & \textit{for } n \geq 0 \\ 0 & \textit{elsewhere.} \end{cases} \tag{8}$$

Prove that the recursive equation that describes the output $y[n]$ of this filter as a function of the input $x[n]$ is

$$y[n] = 0.95y[n-1] + x[n] \quad \textit{for } n \geq 0 \textit{ and } y[-1] = 0. \tag{9}$$

Problem 7: *Using relationship (9) and the `filter` function in MATLAB, generate the random sequence $y[n]$ that results if the input sequence $\{x_n\}$ is filtered by the linear filter. Compute and plot the autocorrelation function $R_x[m]$ and the power spectral density $G_x(f)$ using the following relations:*

$$R_x[m] = \frac{1}{N-m}\sum_{n=1}^{N-m} x[n]x[n+m], \quad \textit{for } m = 0, 1, \ldots 50 \tag{10}$$

$$R_x[m] \overset{\mathcal{F}}{\longleftrightarrow} G_x[k]. \tag{11}$$

Try to plot autocorrelation function $R_x[m]$ and the power spectrum $G_x[k]$ for different ensembles of $x[n]$. It will be noted that the two exhibit a significant variability over different ensembles. To get reasonable results, it is necessary to average the sample autocorrelation and power spectrum over several realizations.

Problem 8: *Compute and plot the autocorrelation function $R_y[m]$ and the power spectral density $G_y(f)$ for the output sequence $\{y_n\}$. Is there any relationship between the input and output power spectral densities? Why or why not? If yes, prove the relationship is indeed valid for the above simulation.*

7. QUADRATURE AMPLITUDE MODULATION:

Problem 9: *Using the code of (2), generate two i.i.d. sequences $\{w_{cn}\}$ and $\{w_{sn}\}$ of $N = 1000$ uniformly distributed random numbers in the interval [-0.5, 0.5]. Compare their respective means and variances with each other and with the theoretical values.*

Problem 10: *Each of these sequences is passed through a linear filter with impulse response*

$$h[n] = \begin{cases} (0.5)^n, & \textit{for } n \geq 0 \\ 0 & \textit{elsewhere.} \end{cases} \tag{12}$$

Prove that the input-output characteristic of the above LTI system, is given by the recursive relation

$$x[n] = 0.5x[n-1] + w[n] \quad \textit{for } n \geq 1 \textit{ and } x[0] = 0. \tag{13}$$

where $x[n]$ is now the output and $w[n]$ the input.

Problem 10: *Calculate the resulting output sequences, $\{x_{cn}\}$ and $\{x_{sn}\}$ when $\{w_{cn}\}$ and $\{w_{sn}\}$ are applied at the input of the system in (b). The output sequence $\{x_{cn}\}$ modulates the carrier $\cos(\pi/2)n$, and the output sequence x_{sn} modulates the quadrature carrier $\sin(\pi/2)n$. The bandpass signal is formed by combining the modulated components as in*

$$x[n] = x_{cn}\cos(\frac{15\pi}{8}n) + x_{sn}\sin(\frac{15\pi}{8}n). \tag{14}$$

Problem 11: *Compute and plot autocorrelation components $R_c[m]$ and $R_s[m]$ for $|m| \leq 10$ for the sequences $\{x_{cn}\}$ and $\{x_{sn}\}$ respectively. Also, compute the autocorrelation function $R_x[m]$ for $|m| \leq 10$ for the bandpass signal $x[n]$. Is there any relationship between the three autocorrelation functions? As in question 2, you are required to average the sample autocorrelations and power spectrums over several realizations.*

Problem 12: *Use the DFT (calculated using* `fft` *function in MATLAB) to compute the power spectra $G_c(f)$, $G_s(f)$, and $G_x(f)$. Plot the power spectra and comment on the results.*

Problem 13: *Design a demodulator to retrieve $\{x_{cn}\}$ and $\{x_{sn}\}$ from $\{x[n]\}$. Implement in MATLAB and show that $\{x_{cn}\}$ and $\{x_{sn}\}$ can be retrieved without any error.*