

PROJECT 2: BASEBAND TRANSMISSION

Instructor: Acknowledgement: This project is developed by Prof. Amir Asif

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1. PURPOSE

In this lab, you will be introduced to transformations of random variables. Several engineering software packages including Matlab include a library function for generating samples of a random variable with uniform distribution. You will learn how transformations can be used to generate samples for a random variable with a different probability density function. You will also implement (i) an uniform pulse code modulator (PCM) that uses a uniform quantizer; and (ii) a nonuniform PCM with a nonuniform quantizer. The nonuniform quantizer is implemented by first distorting the signal using a logarithmic compression characteristic, followed by a uniform quantizer. The logarithmic compression characteristic used in the project is the μ -law compander.

The learners will complete the following three simulations in MATLAB and submit their solutions along with a soft copy of the code in the form of a report.

2. OBJECTIVES

By the end of this project, you will be able to:

1. Generate samples of a random variable for any given probability density function by transforming uniformly distributed random variables.
2. Implement a uniform quantizer used in the pulse code modulation (PCM) system to digitally represent sampled analog signals.
3. Augment the PCM system with a compander used to transmit signals with a large dynamic range over communication channels with smaller dynamic range capability.

3. REFERENCES

1. Bernard Sklar text: Sections 2.6 - 2.7. Pages 79 – 84.

4. TRANSFORMATIONS OF RANDOM VARIABLES

A situation that frequently occurs in the analysis of communication systems is that in which a random variable Y is functionally related to another random variable X whose probability density function (pdf) $p_X(x)$ is known and it is desired to determine the pdf $p_Y(y)$ of the first random variable Y . For example, the pdf of the transmitted signal is known and it is often required to evaluate the pdf of the signal received through a nonlinear channel. A complete discussion of this problem is beyond the scope of this project but a few elementary concepts can be presented and will be useful for subsequent discussions.

In order to formulate the mathematical framework, let the random variable Y be a single-valued, real function of another random variable X , i.e.,

$$Y = g(X) \tag{1}$$

where it is assumed that the pdf $p_X(x)$ of the random variable X is known. Let the pdf of random variable Y be denoted by $p_Y(y)$ which is unknown. For simplicity, it is assumed that $g(X)$ is a monotonically increasing function

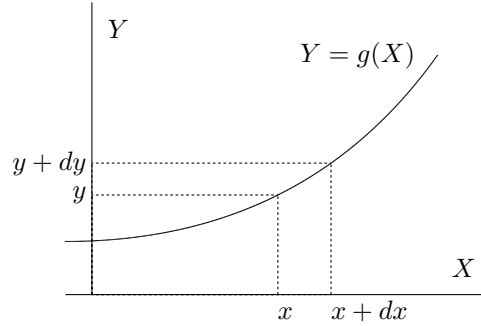


Figure 1: Transformations of variables.

of X , as illustrated in fig. 1. It is clear from fig. 1 that whenever the random variable X lies between x and $x + dx$, the random variable Y will lie between y and $y + dy$. Since the probabilities of these events are equal, therefore,

$$p_Y(y)dy = p_X(x)dx \quad (2)$$

from which the required pdf can be expressed as

$$p_Y(y) = p_X(x) \frac{dx}{dy}. \quad (3)$$

In eq. (3), $g(X)$ is a monotonically increasing function of X . A similar result is obtained when $g(X)$ is a monotonically decreasing function of X . In the later case, the derivative $\frac{dx}{dy}$ will be negative. Since a pdf must be positive, we take the absolute value of the derivative $\frac{dx}{dy}$ in eq. (3). Hence, for either situation

$$p_Y(y) = \left| \frac{dx}{dy} \right| p_X(x) \Big|_{X=g^{-1}(Y)}. \quad (4)$$

The above expression is used to calculate the pdf of any random variable that is a function of a random variable with a known pdf.

Example 1: Consider now a specific example of the transformation of random variables by assuming that the random variable X has the pdf

$$p_X(x) = e^{-x}U(x) \quad (5)$$

where $U(x)$ is the unit step starting at $x = 0$. The random variable Y is related to X by

$$Y = X^3 \quad (6)$$

$$\text{or } X = Y^{1/3} \quad (7)$$

Since y and x are related by the same expression, the derivative $\frac{dx}{dy}$ is given by

$$\frac{dx}{dy} = \frac{1}{3y^{2/3}}. \quad (8)$$

Using eq. (4), the pdf of Y is

$$p_Y(y) = \frac{1}{3}y^{-2/3}e^{-y^{1/3}}U(y). \quad (9)$$

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5. COMPUTER GENERATION OF RANDOM VARIABLES

A digital computer is often used to simulate systems in order to estimate their performance with noise prior to the actual construction of the system. These simulations usually require that random numbers be generated that are values of random variables having prescribed distributions. If software exists for the specified distribution, there is no problem. However, if a computer library does not contain the desired program, it is necessary for the simulation to generate its own random variables. Here we briefly describe how to generate a random variable with any known distribution, given mainly that the computer is able to generate random numbers that are values of a random variable with uniform distribution on $(0,1)$. In Matlab, the function `rand` generate random numbers with uniform distribution. The problem then is to find the transformation

$$Y = g(X) \tag{10}$$

that will create a random variable Y with pdf $p_Y(y)$ when X has the pdf $p_X(x)$. Both pdfs $p_Y(y)$ and $p_X(x)$ are therefore known. Since X has uniform distribution, $p_X(x) = 1$ for $(0 \leq x \leq 1)$.

Referring to fig. 1, the probability of the event $\{Y \leq y\}$ equals the probability of the event $\{X \leq x\}$ because of one-to-one correspondence between X and Y , i.e.,

$$P\{Y \leq y\} = P\{X \leq x\} \tag{11}$$

$$\text{or } F_Y(y) = F_X(x) \tag{12}$$

where $F_X(x)$ and $F_Y(y)$ are the distribution functions for the random variables X and Y . For uniform distribution, with $p_X(x) = 1$ for $(0 \leq x \leq 1)$,

$$F_X(x) = x \quad \text{for } 0 < x < 1 \tag{13}$$

therefore, eq. (12) becomes

$$F_Y(y) = x \quad \text{for } 0 < x < 1 \tag{14}$$

from which the expression for y can be evaluated as

$$y = F_Y^{-1}(x) \quad \text{for } 0 < x < 1. \tag{15}$$

The above expression is our principal result. It states that, given a specified distribution $F_Y(y)$ for Y , we find the inverse function by solving $F_Y(y) = x$ for y .

Theoretical Problem 1. (Generation of random numbers with Exponential pdf:) An exponential random variable is defined by the pdf

$$p_Y(y) = \begin{cases} ae^{-ay} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where $a > 0$ is a given constant.

1. Show that the distribution $F_Y(y)$ is given by

$$F_Y(y) = (1 - e^{-ay})U(y). \tag{17}$$

2. Using $F_Y(y)$ and eq. (14), evaluate the relationship $Y = g(X)$ that transform a random variable X with uniform pdf between $(0,1)$ to a random variable Y with exponential pdf.
3. Determine the mean and variance of the exponential random variable defined in (16).

Problem 2. Using the result of Problem 1, generate an independent identically distributed (i.i.d.) sequence $\{y_n\}$ of $N = 10000$ random variables with exponential distribution. Assume $a = 1$. Compute the mean and variance of the generated sequence and compare with the theoretical values obtained analytically from the distribution (16). Plot the histogram of y and show how it relates to the pdf of y in eq. (16).

6. UNIFORM QUANTIZATION

The uniform quantizer is discussed in section 2.5 of the Sklar text. In uniform quantization, the quantization regions are chosen to have equal length, referred to as *quantile interval*. Review section 2.5 of the text and focus on the implementation of a uniform quantizer.

Problem 3. Write a Matlab function `myunfquantizer` with the following calling syntax

```
function [yq] = myunfquantizer(y,n)
```

that quantizes the input vector y having exponential distribution with a uniform quantizer having $L = 2^n$ levels. Take the interval of interest to be $[0, 1]$. The output is returned in the column vector `yq`.

Problem 4. Assuming $a = 1$, quantize the random numbers generated in Problem 2 with uniform quantizers having $L = 2, 4, 8, 16, \dots, 1024$ levels for the source. You may want to scale the random numbers between the range $[0, 1]$ by dividing with the maximum value before applying the uniform quantizer. After quantization, multiply with the maximum value to restore the original scale.

Problem 5. Compute the quantization error (QE) between the original vector y and the quantized vector \hat{y} defined as

$$\text{QE} = \sum_{i=1}^N (y(i) - \hat{y}(i))^2 \quad (18)$$

for $L = 2, 4, 8, 16, \dots, 1024$. Note $\hat{y}(i) = \text{yq}$ in your case and i is the index for the samples.

Problem 6. Plot the entropy of the quantized source and $\log_2(L)$ as function of n on the same figure. Comment on your results. Recall entropy $H(y) = -\sum_{i=1}^L \text{yq}(i) \Pr(\text{yq}(i))$ where $\Pr(\text{yq}(i))$ denotes the probability (normalized frequency of occurrence) of the quantized sample $\text{yq}(i)$ set to level i , for $(1 \leq i \leq L)$.

7. NONUNIFORM QUANTIZATION

Nonuniform quantization can be achieved by first passing the input signal through a nonlinear system that reduces its dynamic range and then applying the input to a uniform quantizer. At the receiving end, the output is passed through the inverse of the nonlinear element used in the transmitter. Here we will use the μ -law characteristics

$$z = \frac{\log(1 + \mu|y|)}{\log(1 + \mu)} \text{sgn}(y) \quad (19)$$

where y is the normalized input ($|y| \leq 1$) and μ is a positive constant. The inverse of the μ -law nonlinearity is

$$y = \frac{(1 + \mu)^{|z|} - 1}{\mu} \text{sgn}(z) \quad (20)$$

Problem 7. Derive equation (20) from (19).

Problem 8. Plot the input-output characteristics of the nonuniform quantizer obtained from (19) using $\mu = 255$.

Problem 9. Repeat Problem 4 for the random variables y that have been compressed using (19). In other words, generate random variables y with exponential distribution, compress them using (19), and then use uniform quantization to quantize them. Compare your results with the results of Problem 4 and comment on the differences.

Problem 10. List the first 32 samples of the random variable y with exponential pdf, the output of the uniform quantizer in Problem 4, and the output of Problem 9. Make sure that you use the same ensembles of y in each case. Comment on your results.

Problem 11. Using equation (20), expand the output of Problem 9. Compare the first 32 samples with the first 32 samples of the output of the uniform quantizer in Problem 4

Problem 12. Repeat Problems 9-11 for two values of μ (other than 255) in the range $0 \leq \mu \leq 255$.