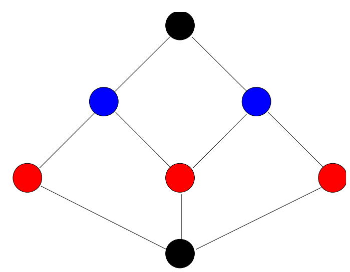
Conceptual Granularity, Fuzzy and Rough Sets



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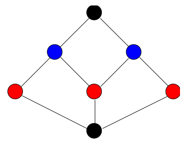


Outline

1. Introduction: Frames and Granularity
2. Conceptual Scaling Theory
3. Conceptual Interpretation of Fuzzy Theory
4. Conceptual Interpretation of Rough Set Theory

Frames

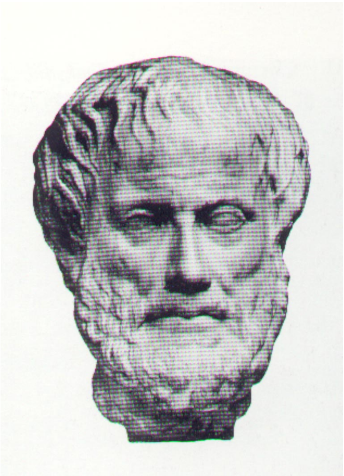
- in art: frame of a painting

- in geometry: coordinate system
- in knowledge processing:

  context for the embedding of information

- refinement of frames leads to a finer granularity

Precision and Granularity



Aristotle

Aristotle

(Physics, book VI, 239a, 23):

During the time when a system is moving, not only moving in some of its parts,

it is impossible that the moving system is precisely at a certain place.

Einstein's Granularity Remark

Albert Einstein:

„Zur Elektrodynamik bewegter Körper"

Annalen der Physik **17** (1905): 891-921

Footnote on page 893:

„Die Ungenauigkeit, welche in dem Begriff der Gleichzeitigkeit zweier Ereignisse an (annähernd) demselben Orte steckt und gleichfalls durch eine Abstraktion überbrückt werden muß, soll hier nicht erörtert werden."

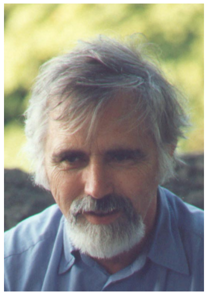
Granularity in Knowledge Representations

- Statistics
- Clusteranalysis
- Interval Mathematics
- Spatio-Temporal Granularity (Robotics)
- Granularity Reasoning

Recent Granularity Theories and their Founders

Lotfi Zadeh: Fuzzy Theory (1965)

Zdzislaw Pawlak: Rough Set Theory (1982)



Rudolf Wille: Conceptual Scaling Theory (1982)

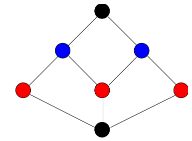
Second International Conference on Rough Sets and Current Trends in Computing,
Banff /Kanada, 16.-19.10.2000.

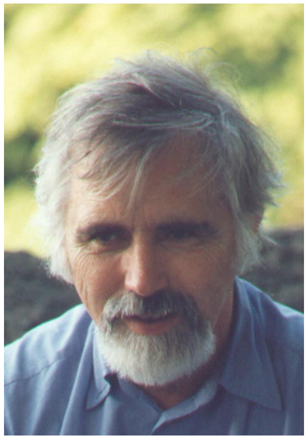
Ziarko Pawlak Wolff

Zadeh

Skowron

Conceptual Knowledge Processing





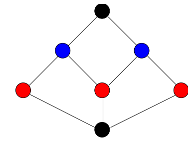
Rudolf Wille

- Formal Concept Analysis 1982
- Mathematizing the concept of „concept“:
- Visualization of conceptual hierarchies

- Data Analysis
- Conceptual Scaling Theory
- Conceptual Knowledge Acquisition

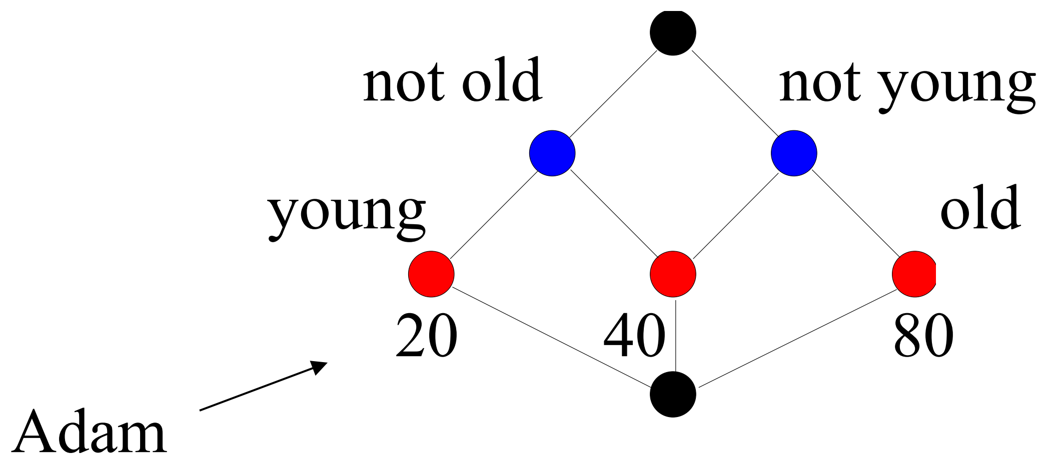
- Contextual Logic
- Conceptual Relational Structures
- Temporal Concept Analysis

Conceptual Scaling



- Main application: Data Analysis
- Main idea: Embed objects into conceptual frames
- Conceptual frames: Formal contexts describing the values

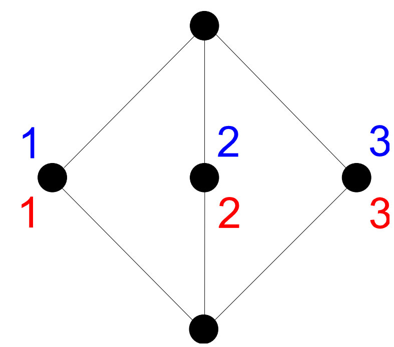
	age
Adam	20
Bill	40
Chris	80



Examples of Scales (1)

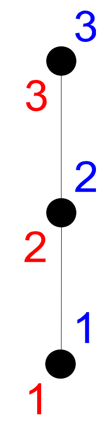
- Nominal scales:

=	1	2	3
1	×		
2		×	
3			×



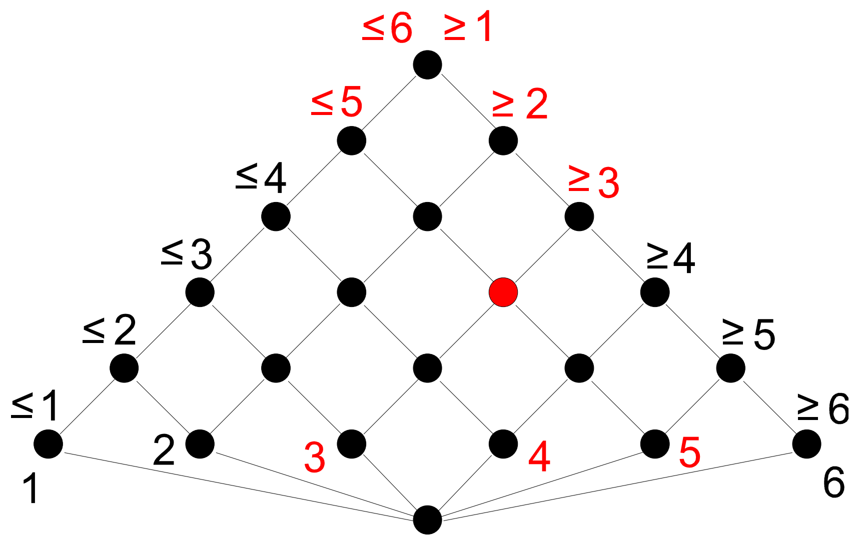
- Ordinal scales:

≤	1	2	3
1	×	×	×
2		×	×
3			×



Examples of Scales (2)

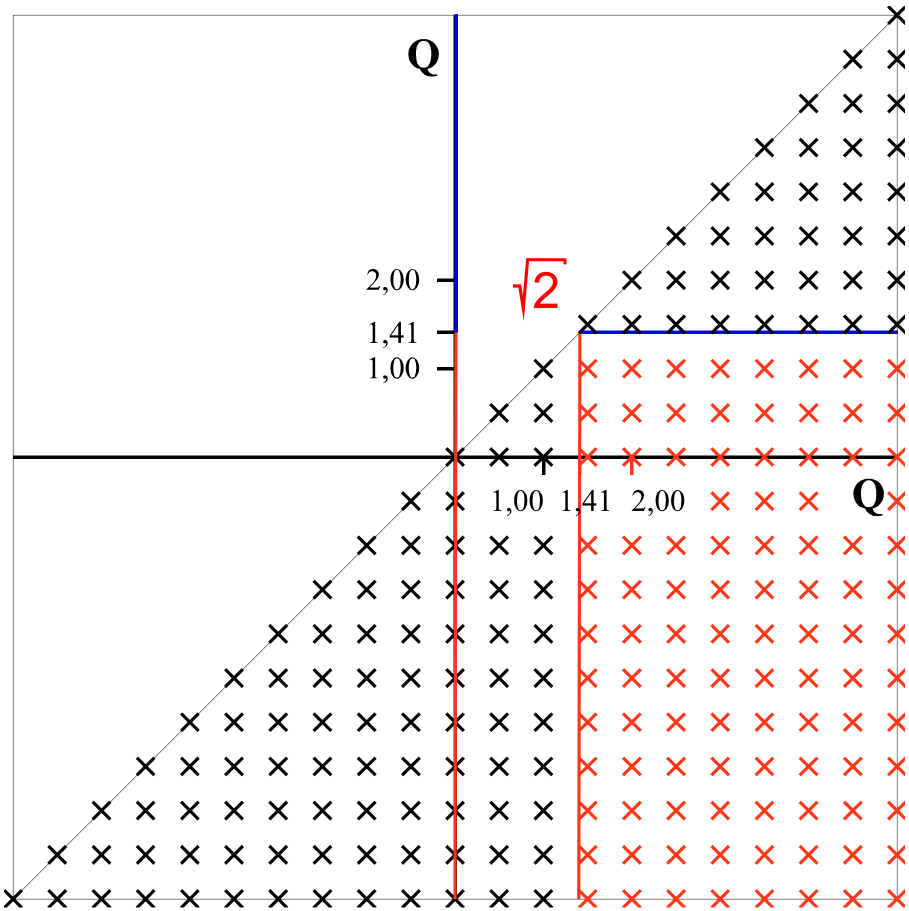
Interordinal scales:



[3,5] = {x| 3 ≤ x ≤ 5 }

Examples of Scales (3)

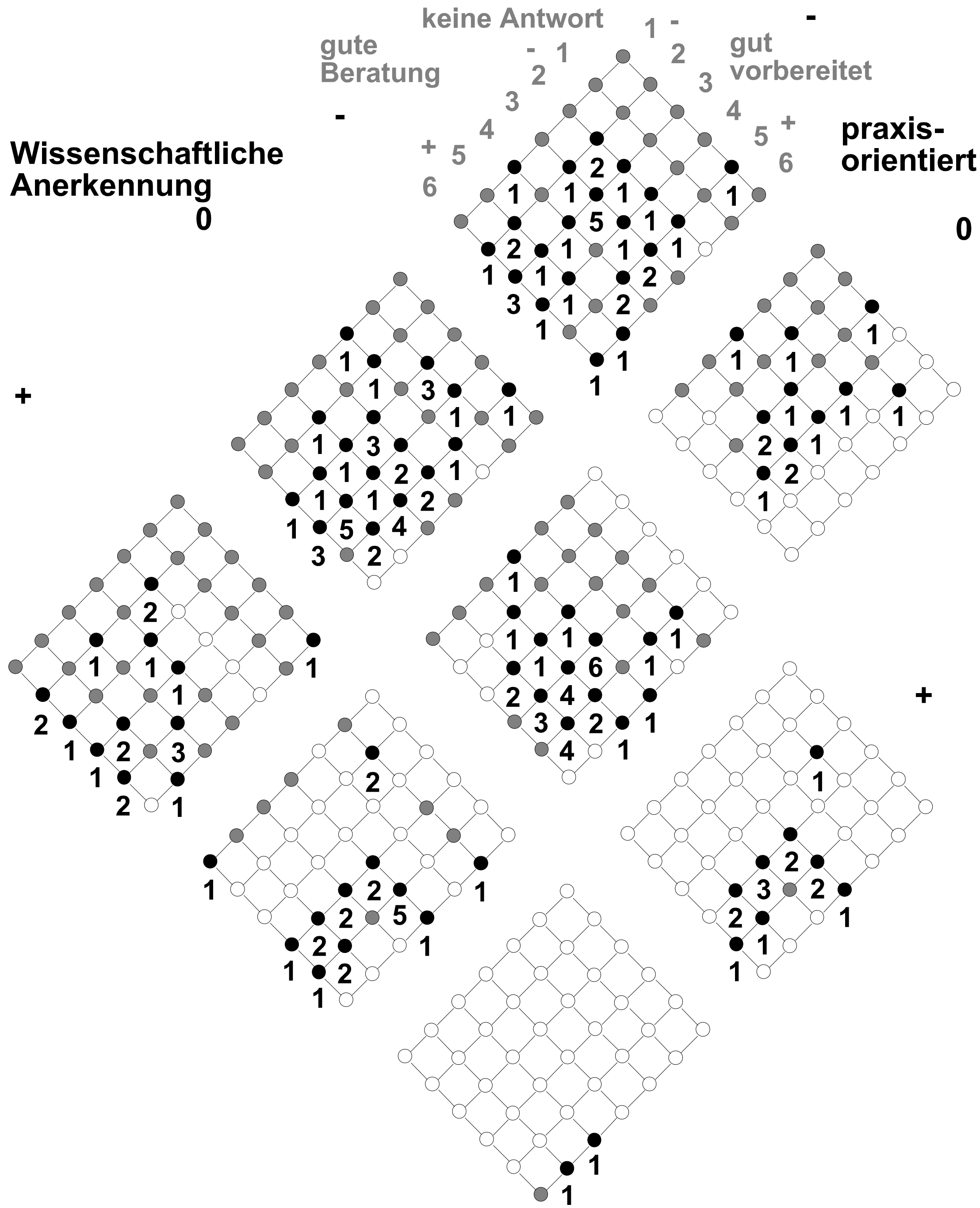
The definition of real numbers as concepts of a formal context:



R := **B**(Q,Q,≤) \ {[symbol],-[symbol]}

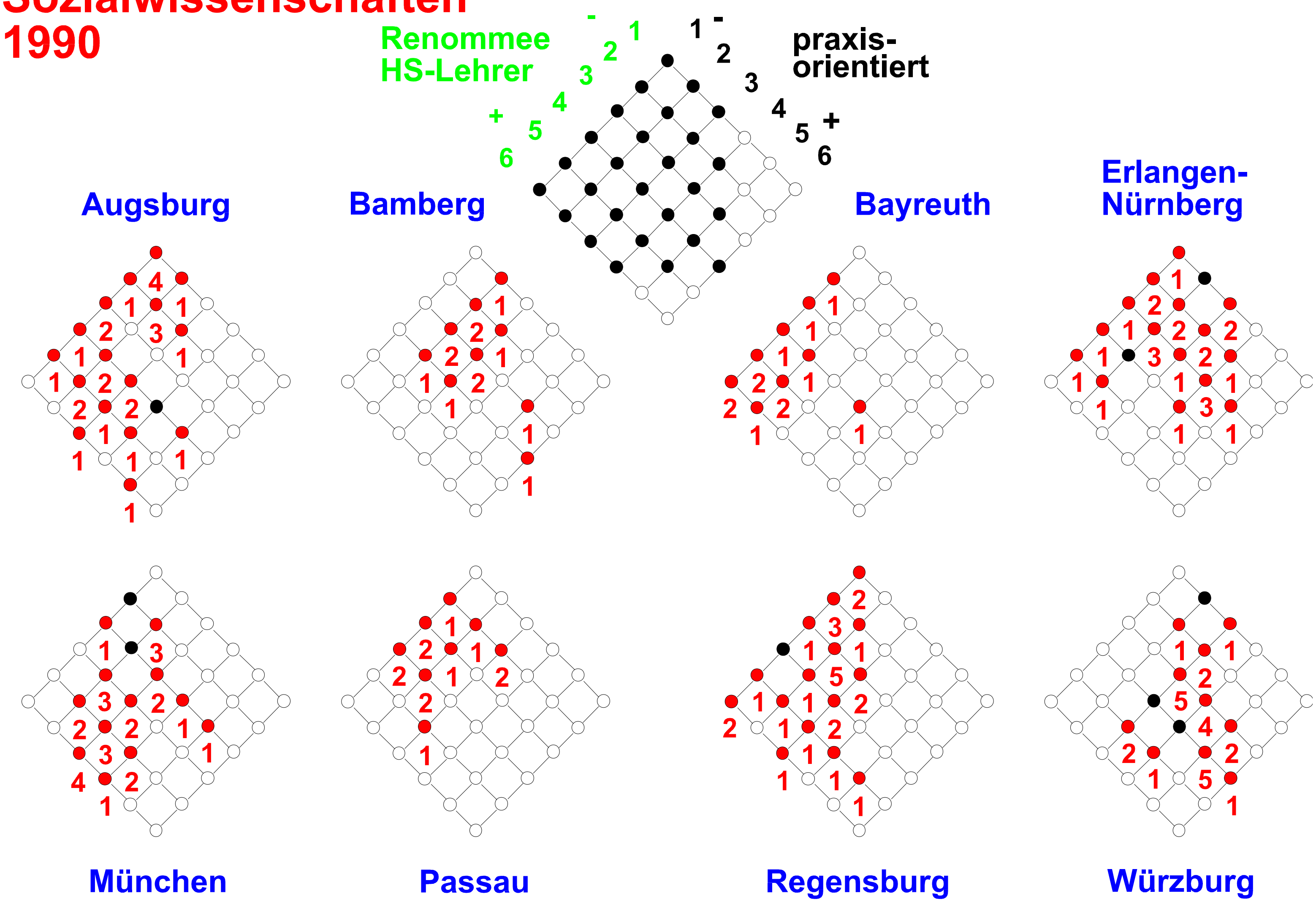
[symbol]:= (Q,∅)

-[symbol]:= (∅,Q)


keine Antwort
gute Beratung
gut vorbereitet
Wissenschaftliche Anerkennung
praxis-orientiert

Bayerische Universitäten
Sozialwissenschaften
1990

NBJSCBI



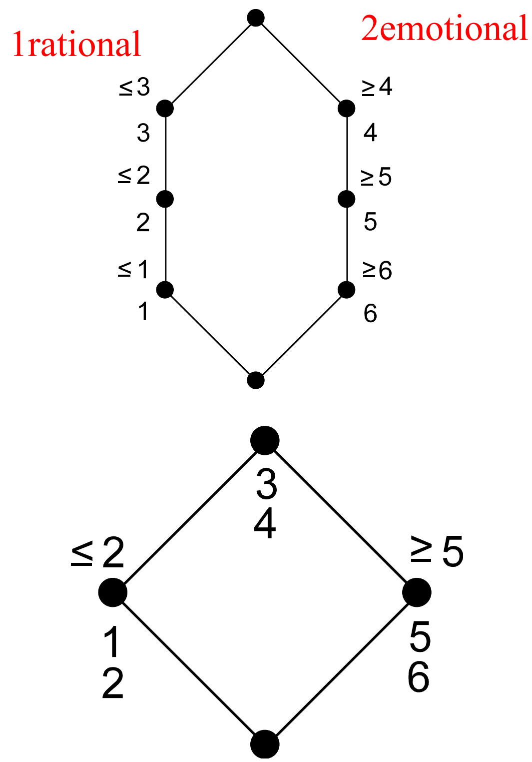
Applications of Scales (1)

Data of an Anorectic Young Woman:

1rational 2emotional

1 2 3 4 5 6

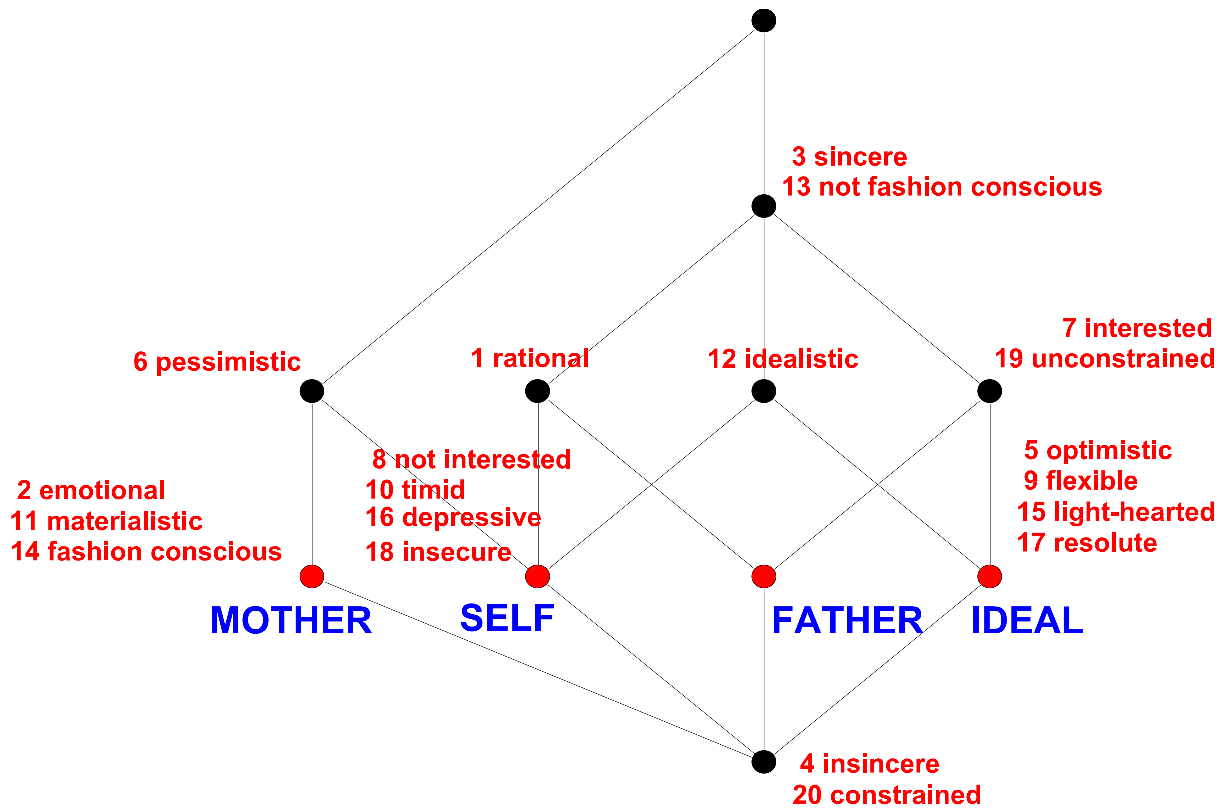
many-valued	1rat-2emot
SELF	2
MOTHER	5
...	



derived	1rat	2emot
SELF	×	
MOTHER		×
...		

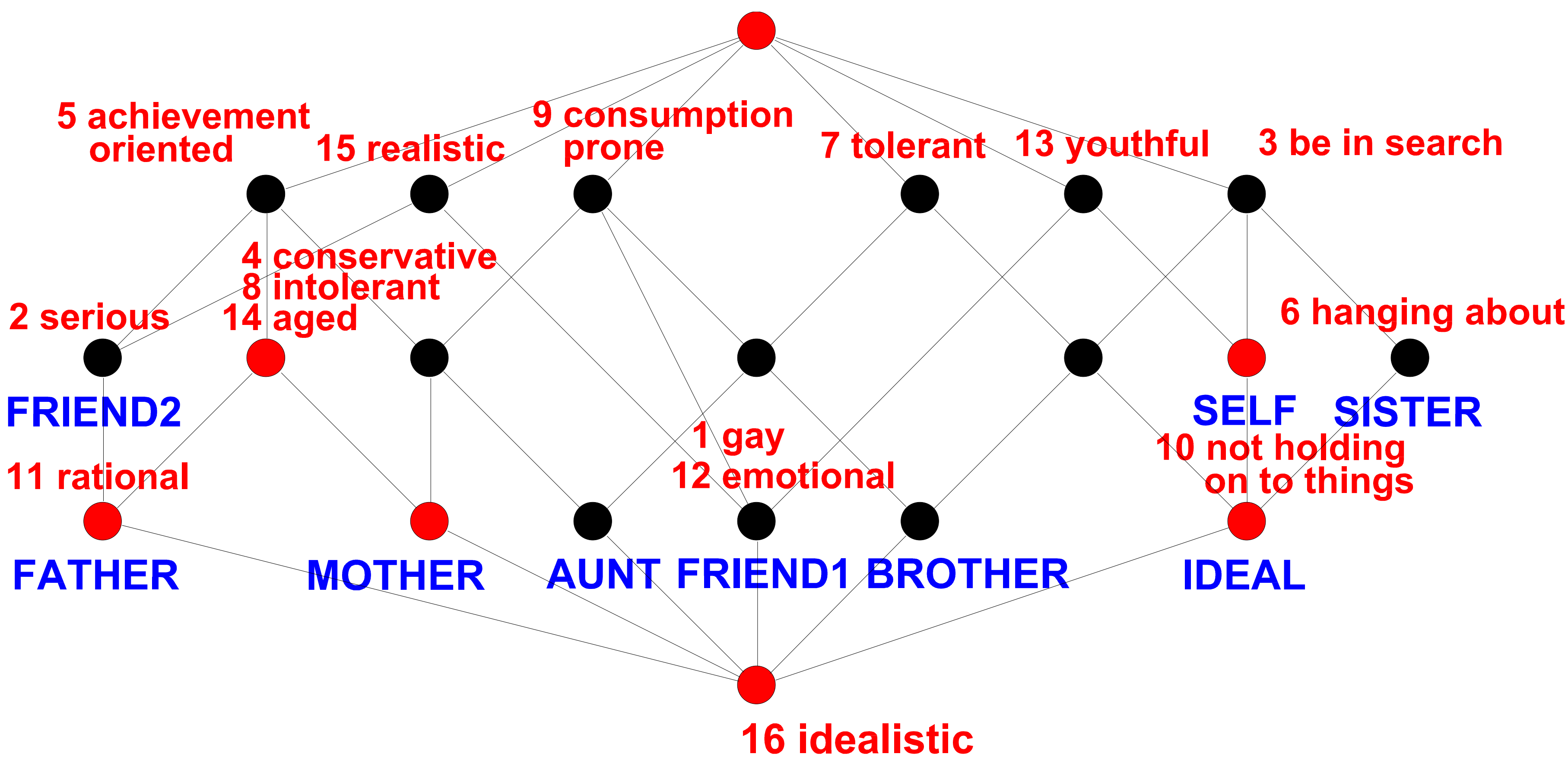
Applications of Scales (2)

Data of an Anorectic Young Woman:
Beginning of treatment



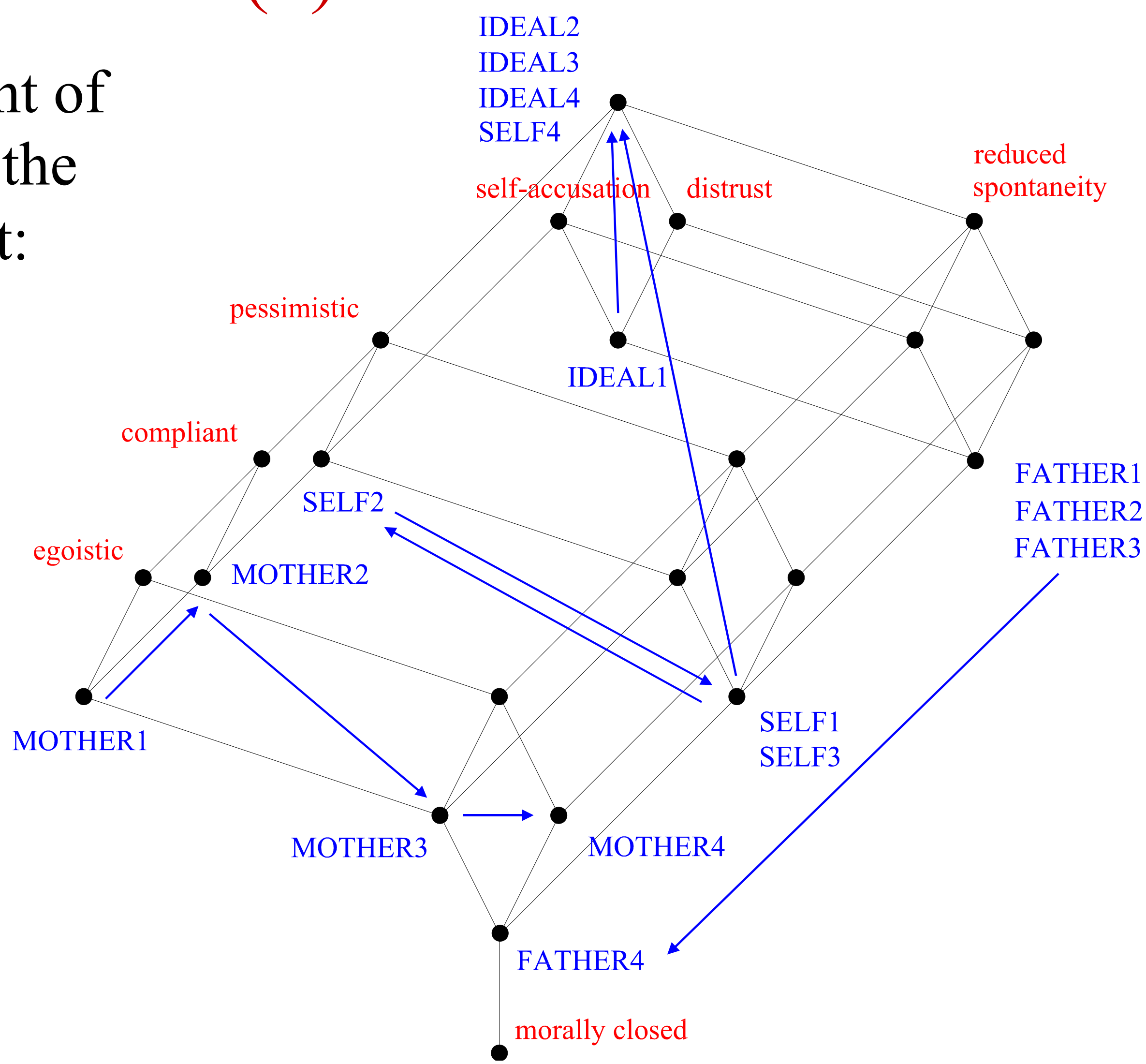
Applications (3)

Data of an Anorectic Young Woman: End of treatment

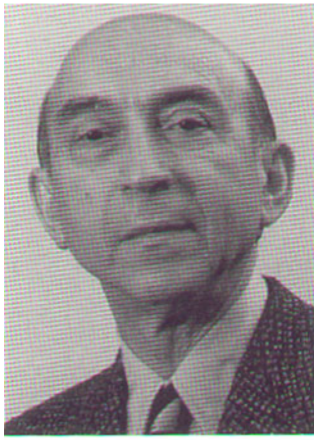


Applications (4)

The point of view of the therapist:



Conceptual Interpretation of Fuzzy Theory

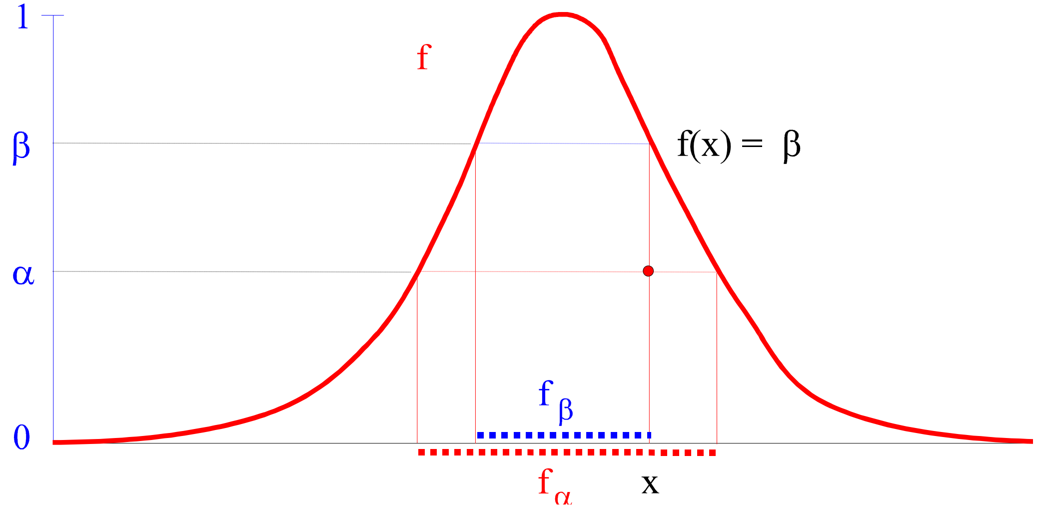


Lotfi A. Zadeh

Lotfi A. Zadeh (1965):
Fuzzy Theory:
„theory of graded concepts"
„in which everything is a matter of degree
or to put it figuratively,
everything has elasticity."

1995 IEEE Medal of Honor

Membership Function (Fuzzy Set)

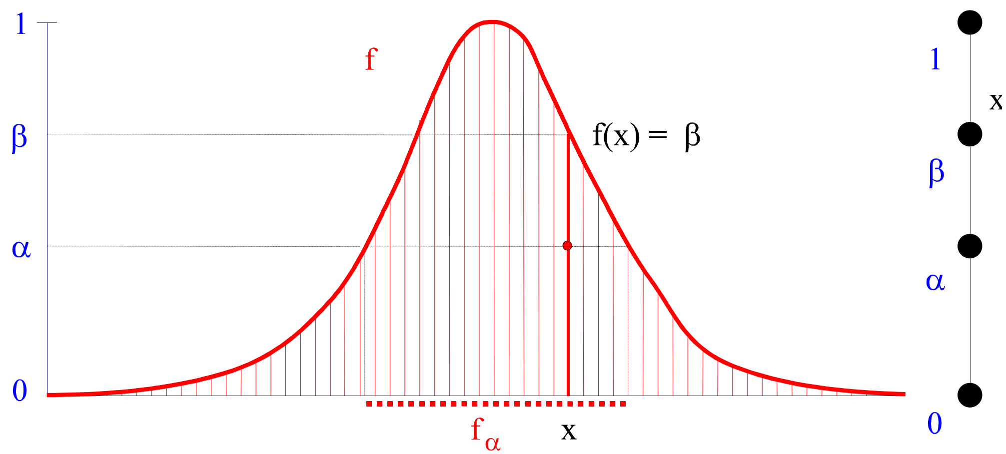


Def.: Let X be a set and f: X → [0,1]. Then f is called a membership function (or a fuzzy set) on X.

„graded concepts“ are described by the linear order on [0,1]

There is no formal object representation in Fuzzy Theory!

The cut-context of a Fuzzy set



Def.: The cut-context of a membership function $f: X \rightarrow L$

$K_f := (L, X, I_f)$

where $\alpha\ I_f\ x :\Leftrightarrow f(x) \geq \alpha$.

Lemma: The concept lattice of the cut-context is a chain which determines f uniquely.

Linguistic Variables:

Zadeh (1975):

“By a *linguistic variable* we mean a variable
whose values are words or sentences
in a natural or artificial language.

For example, *Age* is a linguistic variable
if its values are linguistic rather than numerical,
i.e., *young, not young, very young, quite young, old, not very old and not very young,* etc.,
rather than 20, 21, 22, 23,....”

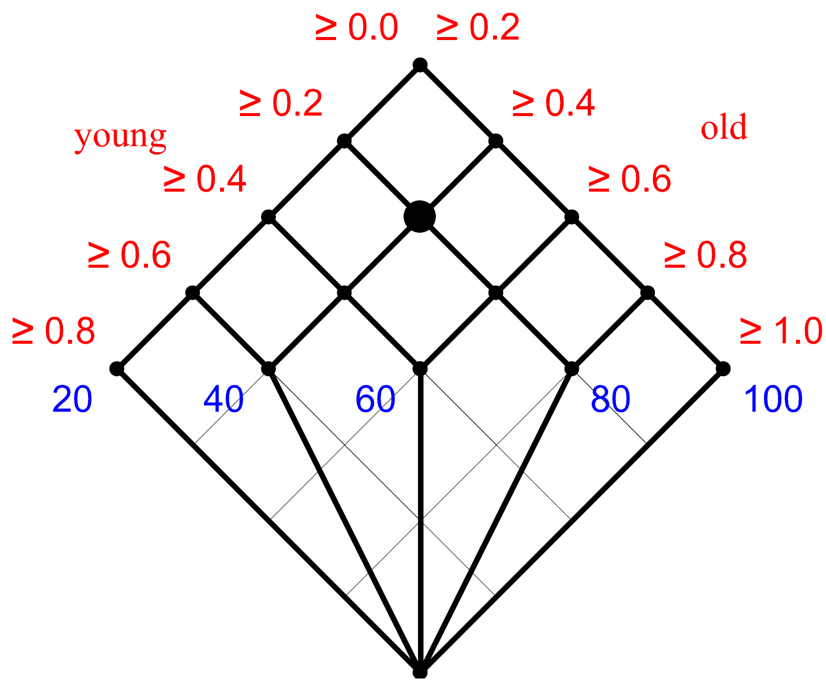
Linguistic Variables: Example

X	young	old
20	0.8	0.2
40	0.6	0.4
60	0.4	0.6
80	0.2	0.8
100	0	1.0

Scaling the membership values!

The Context of a Linguistic Variable:

X	young					old				
	0.0	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8	1
20	×	×	×	×	×	×				
40	×	×	×	×		×	×			
60	×	×	×			×	×	×		
80	×	×				×	×	×	×	
100	×					×	×	×	×	×

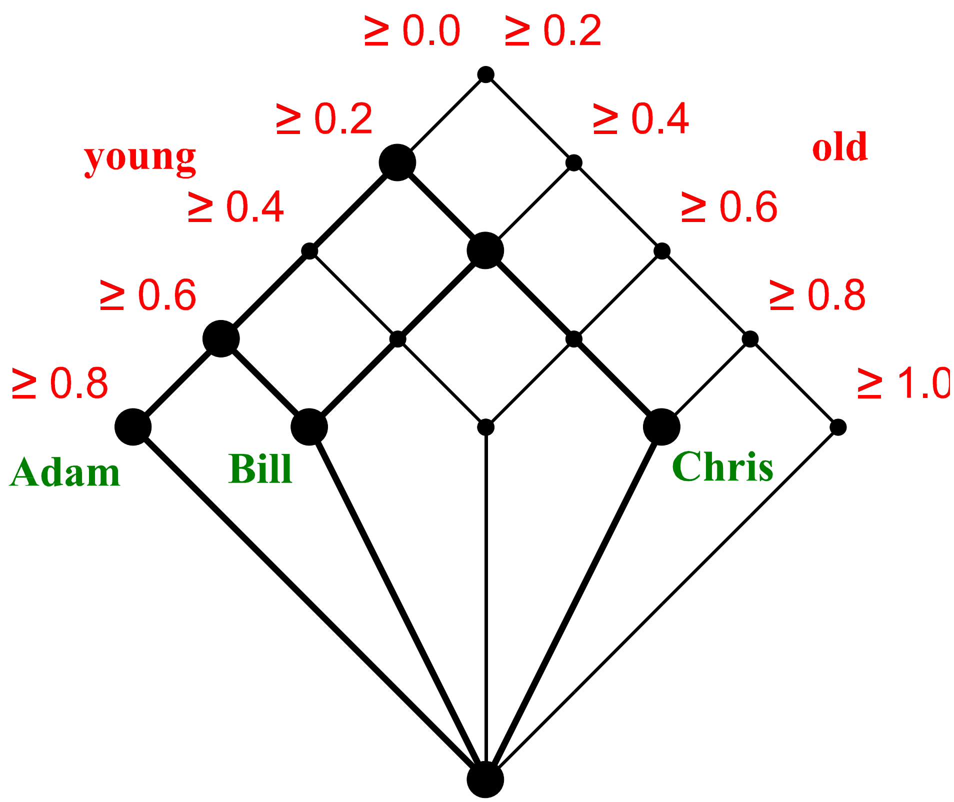


The Realized Scale: „If an object comes in...“

	age
Adam	20
Bill	40
Chris	80

Scaling the age values!

Second scaling!



L-Fuzzy Sets for an ordered set (L,≤)

Definition:
Let X be a set and (L, ≤) an ordered set.

$F(X,L) := \{ f \mid f: X \rightarrow L \}$

is called the set of all *L-Fuzzy sets*
(or L-membership functions) on X.

The cut-context of an L-Fuzzy set is defined in the same way as for classical Fuzzy sets.

Definition: The product of two L-Fuzzy sets
Let $f \in F(X,L)$, $f' \in F(X',L')$

$(f \times f')(x, x') := (f(x), f'(x')) \in L \times L'$

$f \times f' \in F(X \times X', L \times L')$.

$(L \times L', \leq_{\times})$

is the usual product order.

Linguistic Variables over an Order Set $(L,\leq)$

Definition:

A linguistic variable (over an ordered set $(L, \leq)$)
is a quintupel $(X, V, \boldsymbol{\mu}, L, \leq)$,
where X is a set (called the domain),
V is a set (of linguistic values),
$(L, \leq)$ is an ordered set and
$\boldsymbol{\mu}: V \rightarrow F(X, L)$ is a mapping
which represents each linguistic value v by an L-Fuzzy set $\boldsymbol{\mu}_v := \boldsymbol{\mu}(v)$ on X.

X	young	old
20	0.8	0.2
40	0.6	0.4
60	0.4	0.6
80	0.2	0.8
100	0	1.0

Now with values in L!

Realized Linguistic Variables over an Ordered Set $(L,\leq)$

Definition:
Let $\lambda = (X, V, \boldsymbol{\mu}, L, \leq)$ be a linguistic variable,
G a set (of "objects") and
m: $G \rightarrow X$ (a "measurement").
Then
$\mathbf{(G,m,\lambda)} := \mathbf{(G,m,} X, V, \boldsymbol{\mu}, L, \leq\mathbf{)}$
is called a realized linguistic variable.

Products of Realized Linguistic Variables over an Ordered Set $(L,\leq)$

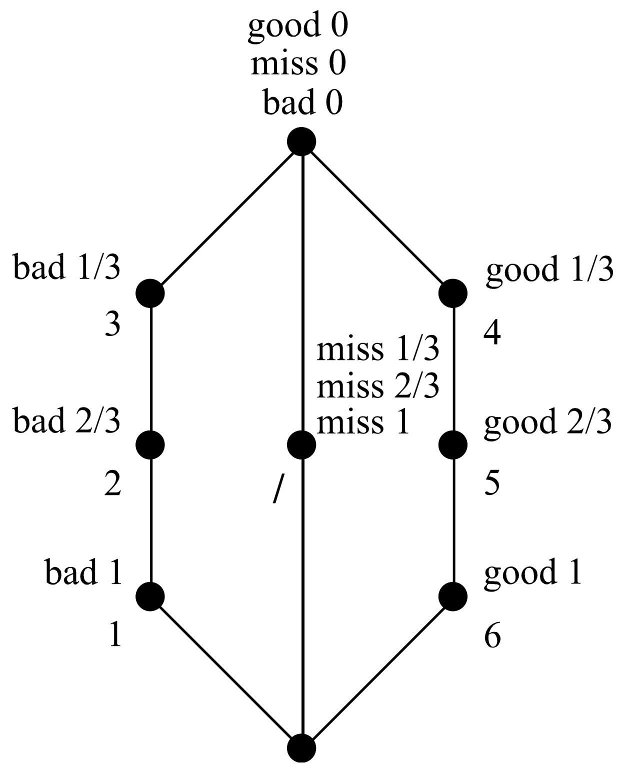
Let $\boldsymbol{\rho} := (G, m, X, V, \boldsymbol{\mu}, L, \leq)$ and $\boldsymbol{\rho}' := (G, m', X', V', \boldsymbol{\mu}', L', \leq')$ be two realized linguistic variables on the same set G of objects.
The mapping $m \times m' : G \to X \times X'$ which is defined by $(m \times m')(g) := (m(g), m'(g))$ is called the *product of the two measurement functions m and m′*.

The mapping $\mu \times \mu' : V \times V' \to F(X \times X', L \times L')$ is defined by
$(\mu \times \mu')(v, v') := \mu_v \times \mu'_{v'}$, where
$(\mu_v \times \mu'_{v'})(x, x') := (\mu_v(x), \mu'_{v'}(x'))$.

Then the following tuple
$\boldsymbol{\rho} \times \boldsymbol{\rho}' := (G, m \times m', X \times X', V \times V', \boldsymbol{\mu} \times \boldsymbol{\mu}', L \times L', \leq_\times)$
is a realized linguistic variable on the product $(L \times L', \leq_\times)$, called the *product of* ρ *and* ρ'.
$\boldsymbol{\lambda} \times \boldsymbol{\lambda}' := (X \times X', V \times V', \boldsymbol{\mu} \times \boldsymbol{\mu}', L \times L', \leq_\times)$ is called the *product of the corresponding linguistic variables* $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$.

An L-Fuzzy Linguistic Variable

with two membership functions: good, bad, and a missing value



Problems in classical Fuzzy Theory

For two classical linguistic variables over [0,1]
their product is no longer a classical linguistic variable
since the direct product

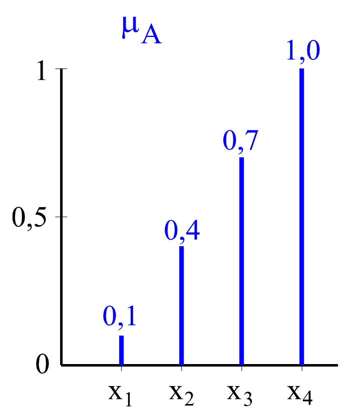
- $[0,1] \times [0,1]$ is not a chain!

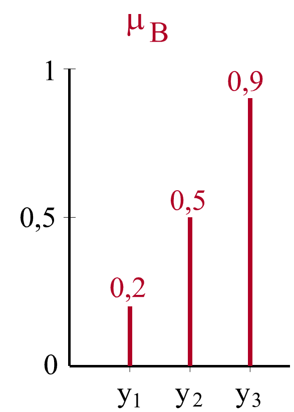
Hence in classical Fuzzy Theory
the direct product of linguistic variables
- can not be defined!

That and the
- missing object representation

is the reason why so many people
did not succeed in defining object based Fuzzy implications
(Gaines-Rescher, Goguen, Gödel, Larsen, Lukasiewicz,
Kleene-Dienes, Mamdani, Reichenbach, Zadeh).

The Mamdani Implication





Min(blue, red)

		y_1	y_2	y_3	
		0,2	0,5	0,9	$\mu_B(y)$
x_1	0,1	0,1	0,1	0,1	
x_2	0,4	0,2	0,4	0,4	
x_3	0,7	0,2	0,5	0,7	
x_4	1,0	0,2	0,5	0,9	
$\mu_A(x)$					

If blue is big and Min(blue, red) is big, then red is big.

Conceptual Interpretation of Rough Set Theory (RST)

Z. Pawlak:
Rough Sets: Theoretical Aspects of Reasoning About Data.
Kluwer Academic Publishers, 1991.
page 3:
“We will be mainly interested in this book with concepts which form a partition (classification) of a certain universe U...".

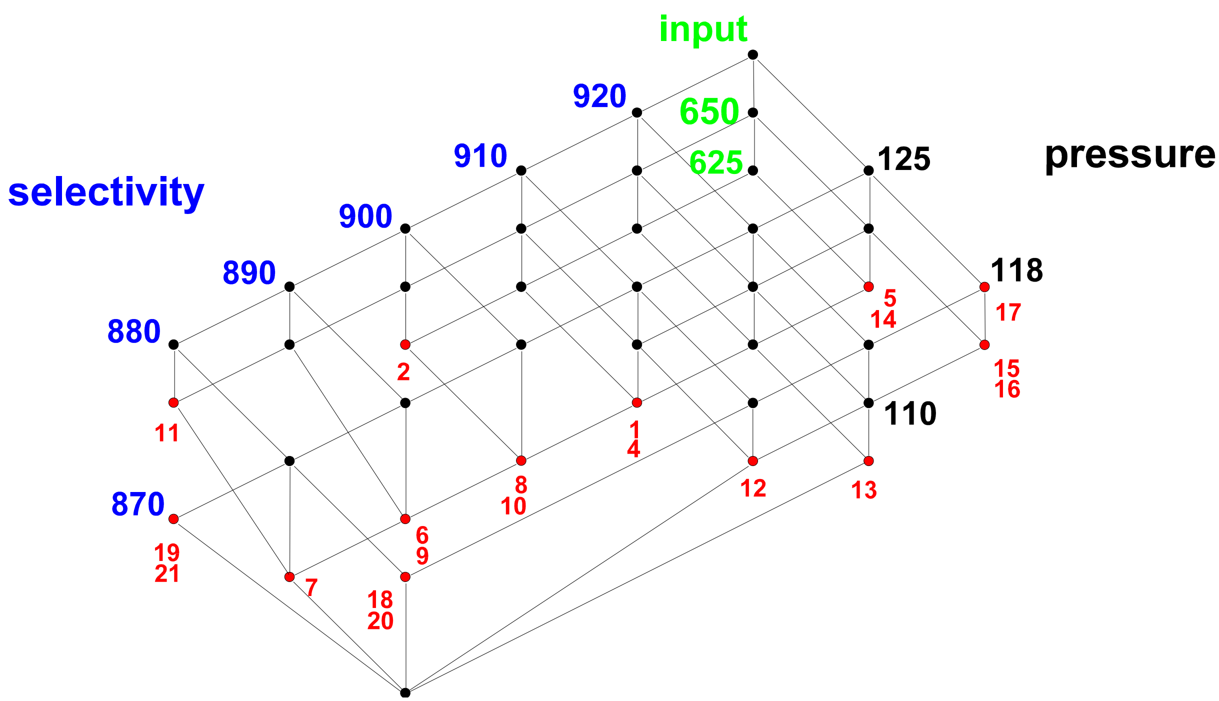
Each partition yields a nominal scale and vice versa.

The notion of “concept” in RST is mainly used extensionally, namely as a subset of the universe U.

Indiscernibility and Contingents

Two objects are indiscernible in the sense of Rough Set Theory

iff they have the same object concept.



Knowledge Bases
in Rough Set Theory

Definition: (Pawlak, Rough Sets, p.3)
A familiy of classifications over U will be called a knowledge base over U.

We describe a knowledge base by a scaled many-valued context

((G,M,W,I), ($\mathbf{S}_m$ | m ∈ M)) using nominal scales.

Theorem 1:

Let (U,**R**) be a knowledge base. Then the scaled many-valued context **sc**(U,**R**) := ((U,**R**,W,I), ($\mathbf{S}_R$ | R ∈ **R**)) is defined by: W:= { $[x]_R$ | x ∈ U, R ∈ **R**} and (x,R,w) ∈ I :⇔ w = $[x]_R$ and the nominal scale $\mathbf{S}_R$ **:=** (U/R, U/R, =) for each many-valued attribute R ∈ **R**. Then the indiscernibility classes of (U,**R**) are exactly the contingents of the derived context **K** of **sc**(U,**R**).

Knowledge Bases
and Scaled Many-Valued Contexts

Theorem 2:
Let **SC**:= ((G,M,W,I), ($\mathbf{S}_m$ | m ∈ M)) be a scaled many-valued context, and **K** := (G, {(m,n) | m ∈ M, n ∈ M_m }, J) its derived context. Then the knowledge base **kb**(**SC**) is defined by **kb**(**SC**):= (G, **R**), where **R** := {R_m | m ∈ M} and for m ∈ M R_m := {(g,h)∈ G×G | $\gamma_m(g) = \gamma_m(h)$ } and γ_m is the object-concept mapping of the m-part of **K**; clearly, the m-part of K is the formal context (G, {(m,n) | n ∈ M_m }, J_m) where J_m := {(g,(m,n)) ∈ J | n ∈ M_m }. Then the indiscernibility classes of **kb**(**SC**) are exactly the contingents of the derived context **K** of **SC**.

Theorem 3:
For any knowledge base (U, **R**): **kb**(**sc**(U, **R**)) = (U, **R**).

Thank you!

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http://www.fbmn.fh-darmstadt.de/home/index.htm