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Ten claims about fuzzy logic

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Here I present, in a more or less telegraphic way, ten claims that I advocated at the Springer forum. Support for the claims may be found in the works listed in the references.[1]

1. *Fuzzy logic can be developed as a fully-fledged formal logical system. But some distinctions are absolutely necessary.*

2. *Distinguish between fuzzy logic in wide and narrow sense* (Zadeh: FLw, FLn). FLw as everything dealing with fuzziness; FLn - a logical system based on many-valued logic. Zadeh stresses that FLn is based on many-valued logic but has a different agenda - e.g. fuzzy modus ponens, usuality quantifiers... My comment to this "but"is: These things also underlie formal logical analysis.

3. *Distinguish between fuzziness as impreciseness* (*vagueness*) *and uncertainty as beliefs. The former deals with degrees of truth, the latter with degrees of* (*un*) *certainty*. The main reason consists in the following: FL may be (and mostly is) truth-functional; logics of belief are *not*. FL is a many-valued logic, logic of belief is a kind of modal logic. Caution: non-truth functional FLn is also possible.

4. *Be explicit about your set of truth values and decide if your logic is built as truth-functional*. You may have finitely/infinitely many truth values, linearly ordered or not. The most frequent (and very fruitful) approach is to concentrate to the unit interval [0,1] of reals as the set of truth values, with its natural ordering, but to study it in the context of an appropriate class of algebras of truth values, not necessarily linearly ordered.

5. *Take a t-norm $*$ as conjunction*; *this defines implication as its residuum and negation as* "*x implies falsity*". Thus the truth function of implication satisfies $x \Rightarrow y = \max\{z | x * z \leqslant y\}$. There is a simple axiom system (BL - basic logic) sound for all t-norm logics and complete with respect to truth in each BL-algebra (regular residuated lattice). The corresponding bazic fuzzy predicate logic is also completely axiomatized by a natural and simple axiom system.

The following formulas are axioms of the basic propositional logic:

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \tag{A1}$$

$$(\varphi \,\&\, \psi) \to \varphi \tag{A2}$$

$$(\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi) \tag{A3}$$

$$(\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi)) \tag{A4}$$

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi) \tag{A5a}$$

$$((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi)) \tag{A5b}$$

$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \tag{A6}$$

$$\bar{0} \to \varphi \tag{A7}$$

Note that the min-conjunction and max-disjunction are definable:

$\varphi \wedge \psi$ is $\varphi \,\&\, (\varphi \to \psi)$,

$\varphi \vee \psi$ is $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

6. *Particular t-norms give particular logics* (*Łukasiewicz, Gödel, product*) *with well elaborated logical properties – also for the corresponding predicate calculus.*
Łukasiewicz (twenties)

$(x * y) = \max(0, x + y - 1)$

$(x \Rightarrow y) = \min(1, 1 - x + y)$

$\neg x = 1 - x$

Gödel (1932)

$x * y = \min(x, y)$

$x \Rightarrow y = 1$ for $x \leqslant y$

y for $x > y$

$\neg x = 0$ for $x > 0$

$\neg 0 = 1$

Product

$x * y = x \cdot y$

$x \Rightarrow y = 1$ for $x \leqslant y$

y/x for $x > y$

$\neg$ as Gödel

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7. *FLn – pure logicians should care*. Fuzzy logic in the narrow sense is a fully fledged symbolic systems with (generalized) Tarskian semantics, with many results on (non) axiomatizability, computational complexity. There are several fascinating open problems. Philosophical logicians might be interested in an analysis of the notion of vagueness and its formalization by many-valued logic (comparative notion of truth).

8. *FLn – AI-community should care*. Till now there is, unfortunately, little communication between the AI community and fuzzy logic community. AI has always stressed its interest in incomplete, vague, uncertain knowledge. Various non-classical logics have become a broadly used means in AI: modal, non-monotonic etc. –why not FL?

AI is in development: after the original stress to symbolic manipulation (non-probabilistic) probability theory (and Dempster-Shafer etc.) has been accepted as theories of uncertainty. It would be a very natural next step to accept (good) fuzzy logic.

9. *FLn – probability theorists should care*. One must clarify the relation of comparative truth and numerical beliefs on (crisp) truth. One possible way is to deal with the fuzzy notion "*A* is probable" and its properties. Further one may study probabilities of fuzzy events (this was initiated by Zadeh).

10. *FLn – soft computing community should care*! Let us repeat: the agenda of FLn can be largely interpreted in terms of *provability* in appropriate symbolic logical systems. IF-THEN rules, so broadly used in fuzzy systems, can be analyzed strictly logically (Gottwald, Kruse and others).

Logical analysis and better knowledge of logic helps to see that e.g. "Mamdani's implication" is a misleading term, which shoud be avoided (with all respect to Mamdani's achievment): minimum is NOT an implication.[2]

Concluding: The logical (formal, symbolic) aspect of fuzzy logic is important and well developed. The reader should feel invited to study it and to contribute to it. Logic (in the narrow sense) gives specific insight – also to soft computing.

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[2] Note in passing that Kosko in his new book "Fuzzy engineering" claims that Kurt Gödel suggested minimum as an implication and calls minimum "Gödel implication operator". This is absolutely false and misleading, to my best knowledge of Gödel's work. Gödel investigated the implication described above (for [0,1] as set of truth values and 1 as truth) and commonly called Gödel implication in his 1932 paper "Zum intuitionistischen Aussagenkalkül".