Rough Mereological Reasoning in Rough Set Theory: Recent Results and Problems

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To the memory of Professor Zdzisław Pawlak

Abstract. This article comes up a couple of months after the death of Professor Zdzisław Pawlak who created in 1982 the theory of rough sets as a vehicle to carry out Concept Approximation and a fortiori, Decision Making, Data Mining, Knowledge Discovery and other activities.

At the roots of rough set theory, was a deep knowledge of ideas going back to Frege, Russell, Łukasiewicz, Popper, and others.

Rough sets owe this attitude the intrinsic clarity of ideas, elegant simplicity (not to be confused with easy triviality), and a fortiori a wide spectrum of applications.

Over the years, rough set theory has been enriched with new ideas.

One of those additions has been rough mereology, an attempt at introducing a regular form of tolerance relations on objects in an information system, in order to provide a more flexible scheme of relating objects than indiscernibility. The theory of mereology, proposed long ago (1916) by S. Lesniewski, proved a valuable source of inspiration. As a result, a more general theory has emerged, still far from completion.

Rough mereology, operating with so called rough inclusions, allows for definitions of a class of logics, that in turn have applications to distributed systems, perception analysis, granular computing etc. etc. In this article, we give a survey of the present state of art in the area of rough mereological theory of reasoning, as we know it, along with comments on some problems.

Keywords: rough sets, granular computing, rough inclusions, rough mereology, granular logics, granular computing, perception calculus, foundations for rough sets.

1 Inexact Concepts: Approximate Reasoning

The case of inexact concepts was discussed by Gottlob Frege (Grundlagen II, 1903) on the margin of his theory of concepts:"..*inexact concepts must have a boundary in which one cannot decide whether the object belongs in the concept or in its complement.. .*" In the realm of mathematics, topology realized this

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idea accurately: around a set not definable in topological terms, i.e., not clopen, there is the nonempty boundary, whose elements have any neighborhood neither in the set nor in its complement. In computer science, this idea was rendered by Professor Pawlak (1982) in his theory of rough sets.

In learning concepts, the obvious prerequisite is to employ a symbolic language for coding objects along with some formulas, i.e.,"knowledge", that form the starting point for attempts at concept description.

1.1 Rough Sets: A Program Envisioned by Zdzisław Pawlak

Let us go back to the idea of a rough set by Zdzisław Pawlak. An abstract setting for this idea, see Pawlak [4], [5] is a pair (U, R), where U is a universe of *objects* and R is an equivalence relation on U (or, for that matter, a family of equivalences on U) called *a knowledge base* (some authors use the term *an approximation space*). The relation R induces a partition into equivalence classes $[u]_R$, interpreted as elementary blocks of knowledge (some say: elementary granules of knowledge).

A practical way of implementing this idea is by using an *information system* [4], i.e., a pair (U, A) where A is a set of *attributes*, each of them a mapping $a : U \rightarrow V$ on U valued in the value set V; the equivalence R is then produced as the indiscernibility relation; $R = IND$ with $uINDv$ iff $a(u) = a(v)$ for each $a \in A$.

An exact concept relative to (U, A) is defined as the union of classes of the relation IND; other concepts are declared *inexact*.

A variant of an information system is a *decision system*, in which one attribute, say d is added, i.e., a decision system is a triple (U, A, d) with $d \notin A$. The decision d represents a classification of objects into decision classes by an external source of knowledge.

Decision logic, see [4], formulates in a logical form dependencies among groups of attributes. Its primitive formulas are descriptors of the form (a, v), where $a \in A \cup \{d\}$ and v a value of a, and formulas are formed by means of propositional connectives $\vee, \wedge, \rightarrow, \neg$. The meaning of a descriptor (a, v) is $[a, v]$=$\{u \in U : a(u) = v\}$, and it is extended recursively to meanings of formulas; in particular, $[p \vee q] = [p] \cup [q]$, $[p \wedge q] = [p] \cap [q]$, $[\neg p] = U \setminus [p]$.

A *decision rule* is a formula of the form $\bigwedge_a (a, v_a) \Rightarrow (d, v)$ that does express a relation between conditional attributes in A and the decision; a set of decision rules is a decision algorithm. In this way rough sets allow for classification and decision solvers.

Concept approximation is achieved by means of rough set approximations; for a concept $X \subseteq U$, the lower, resp., the upper approximation to X is the set, resp., $\underline{A}X = \{u : [u]_A \subseteq X\}$ and $\overline{A}X = \{u : [u] \cap X \neq \emptyset\}$. In this way a concept X is sandwiched between two exact sets. The set $BdX = \overline{A}X \setminus \underline{A}X$ is the boundary of X, in conformity with the Frege idea of sect.1 of the existence of a boundary for inexact concepts.

All these notions have given way to a rich specter of theoretical analysis and application works in the language explained just above.

The question was also: how to enrich the language to absorb many new developments like granular computing, perception calculus and so on? Below we give a subjective view on the status of this question based on some of the author works in years that passed since the year 1997, see, e.g., [15], [8], [9], [10], [11], [12], [13], [14]. Some earlier papers are quoted in the papers mentioned here.

2 Alternative Approaches

Can we have a collective view on concepts that may co–exist with the orthodox, naive–set–theory–based distributive approach exposed above? The answer seems to be "yes".

2.1 A Neoaristotelian Approach: Ontology and Mereology Due to Lesniewski

"Aristotle says in the seventh book of Metaphysics: "If anything were compounded of but one element that one would be the thing itself" (Duns Scotus, Treatise on God as First Principle [18]).

A view contradictory to our set theory. Taken as a principle, it led Stanisław Leśniewski [3] to a new theory of sets (1916) based on the aristotelian notion of part: transitive and non-reflexive relation on nonempty collection of objects. But when the element is defined as a part or the whole object, then each object is an element of itself. Mereology is the theory of collective concepts based on part relation.

Out of distributive concepts, collective concepts are formed by means of the class operator of Mereology.

Mereology is based on the predicate π of part, defined for individual entities, subject to :

P1. $x \pi y \wedge y \pi z \Rightarrow x \pi z$.

P2. $\neg(x \pi x)$.

The element relation el_π induced by π is defined as follows:

$$x\ el_\pi\ y \Leftrightarrow x = y\ or\ x\ \pi\ y.$$

Class of a property M is defined in case a distributive concept M is non–empty; it is subject to,

C1. $x\ \in M \Rightarrow x\ el_\pi\ Cls(M)$.

C2. $x\ el_\pi\ Cls(M) \Rightarrow \exists u, v. u\ el_\pi\ x \wedge u\ el_\pi\ v \wedge v \in M$.

Hence, $Cls(M)$ collects, in one whole object, all objects whose each part has a part in common with an object in M; see remark no. 2 in sect.2.2, below.

2.2 Rough Inclusions

In approximate reasoning mereology works well when diffused to approximate mereology based on the notion of a part to a degree expressed in the form of the predicate $\mu(x, y, r)$ subject to requirements:

RM1. $\mu(x, y, 1) \Leftrightarrow x\ el\ y$.

RM2. $\mu(x, y, 1) \Rightarrow \forall z.[\mu(z, x, r) \Rightarrow \mu(z, y, r)]$.

RM3. $\mu(x, y, r) \wedge s < r \Rightarrow \mu(x, y, s)$.

The relation el is the element relation of the underlying mereology; predicate μ acts on individual objects x, y indicating the degree r to which x is a part of y.

The motivation for this approach can be itemized as follows:

1. Mereology , represented by the predicate el is an alternative theory of sets; rough set theory built on Mereology can be an interesting alternative to traditional rough set theory;
2. Traditional, naive, set theory and Mereology are related: the strict containment $\subset$ is a part relation and $\subseteq$ is the corresponding element relation. In consequence, e.g., for a family of sets F, the class of F is the union of F: $Cls(F) = \bigcup F$.
3. The consequence of the preceding item is that constructs of traditional, naive set – based rough set theory, are a particular case of a more general approach based on a predicate μ – a rough inclusion.

2.3 Rough Inclusions: Specific Definitions

One may ask what form are rough inclusions taking. We consider an information system (U, A) and for $u, v \in U$ we let, $DIS(u, v)$=$\{a \in A : a(u) \neq a(v)\}$, and $IND(u, v) = A \setminus DIS(u, v)$.

Rough inclusions from archimedean t–norms. Consider an archimedean t–norm, i.e., a t–norm $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ with properties that (i) t is continuous; (ii)$t(x, x) < x$ for $x \in (0, 1)$ (i.e., no idempotents except 0,1).

For the norm t as above, a functional representation holds: $t(x, y) = g_t(f_t(x)+ f_t(y))$ with f_t continuous and decreasing automorphism on [0,1], and g_t its pseudo–inverse, see,e.g., [7].

We let, $\mu_t(u, v, r)$ iff $g_t(\frac{|DIS(u,v)|}{|A|}) \geq r$. This defines a rough inclusion μ_t.

Standard examples of archimedean t–norms are : the Łukasiewicz norm $t_L(x, y) = max\{0, x+y-1\}$, and the product (Menger) norm $t_M(x, y) = x \cdot y$.

A justification of probabilistic reasoning. In case of the norm t_L, one has: $f_{t_L}(x) = 1 - x = g_{t_L}(x)$ for $x \in [0, 1]$, hence, $\mu_{t_L}(u, v, r)$ iff $1 - \frac{|DIS(u,v)|}{|A|} \geq r$ iff $\frac{|IND(u,v)|}{|A|} \geq r$.

It is important in applications to have also a rough inclusion on subsets of the universe U; to this end, for subsets $X, Y \subseteq U$, we let, $\mu_{t_L}(X, Y, r)$ iff $g_{t_L}(\frac{|X \setminus Y|}{|U|}) \geq r$ iff $1 - \frac{|X \setminus Y|}{|U|} \geq r$ iff $\frac{|X \cap Y|}{|U|} \geq r$.

The last formula is applied very often in Data Mining and Decision Making as a measure of quality of rules; in rough set decision making, formulas for accuracy and coverage of a rule (see, e.g., Tsumoto's chapter, pp. 307 ff., in [16]) as well as Ziarko's Variable Precision Model approach [20] are based on the probabilistic approach. Similarly, one can apply Menger's t–norm to produce the corresponding rough inclusion.

We restrict ourselves in this article's applications to the Łukasiewicz related t–norms defined above.

The case of continuous t–norms. It is well–known (cf., e.g., [7], [2], papers by Mostert and Shields, Faucett quoted therein) that any archimedean t–norm is isomorphic either to the Łukasiewicz or to the Menger t–norm. Thus, in the realm of archimedean t–norms we have a little choice. Passing to continuous t–norms, it results from the work of Mostert–Shields and Faucett (quoted in [7],[2]) that the structure of a continuous t–norm t depends on the set F of idempotents (i.e, values x such that $t(x,x) = x$); we denote with O_t the countable family of open intervals $A_i \subseteq [0,1]$ with the property that each A_i is free of idempotents and $\bigcup_i A_i = [0,1] \setminus F$. Then, $t(x,y)$ is an isomorph to either t_L or t_M when $x, y \in A_i$ for some i, and $t(x,y) = min\{x,y\}$, otherwise. It is well–known (Arnold, Ling quoted in [7]) that in a representation for min of the form $min(x,y) = g(f(x) + f(y))$, f cannot be either continuous or decreasing.

Rough inclusions from reference objects. We resort to residua of continuous t–norms.For a continuous t–norm $t(x,y)$, the residuum $x \Rightarrow_t y$ is defined as the $max\{z : t(x,z) \leq y\}$. Clearly, $x \Rightarrow_t y = 1$ iff $x \leq y$ for each t.

For an information system (U, A), let us select an object $s \in U$ referred to as a *reference*. For a continuous t–norm t, we define a rough inclusion ν_t^{IND} based on sets $IND(u,v)$, by letting,

$$\nu_t^{IND}(x,y,r) \text{ iff } \frac{|IND(x,s)|}{|A|} \Rightarrow \frac{|IND(y,s)|}{|A|} \geq r. \quad (1)$$

Let us examine the three basic t–norms. In case of t_L, we have: $x \Rightarrow_{t_L} y = min\{1, 1-x+y\}$; thus $\nu_{t_L}^{IND}(x,y,r)$ iff $|IND(y,s)| - |IND(x,s)| \geq (1-r)|A|$.

In case of t_M, we have: $x \Rightarrow_{t_M} y = 1$ when $x \leq y$ and y when $x > y$; hence $\nu_{t_M}^{IND}(x,y,1)$ iff $|IND(x,s)| \leq |IND(y,s)|$ and $\nu_{t_M}^{IND}(x,y,r)$ with $r < 1$ iff $|IND(x,s)| > |IND(y,s)| \geq r \cdot |A|$.

Finally, in case of $t_m = min$, we have $x \Rightarrow_{t_m} y$ is 1 in case $x \leq y$ and $\frac{y}{x}$ otherwise. Thus, $\nu_{t_m}(x,y,r)$ iff $\frac{|IND(y,s)|}{|IND(x,s)|} \geq r$.

Regarding objects x, y as close to each other when $\nu(x,y,r)$ with r close to 1, we may feel some of the above formulas counterintuitive as objects x with "smaller" reference set $IND(x,s)$ may come closer to a given y; a remedy is to define dual rough inclusions, based on the set $DIS(x,s)$ in which case the inequalities in definitions of IND–based rough inclusions will be reverted. In any case, one has a few possibilities here. We state a problem to investigate.

RESEARCH PROBLEM 1. Create a full theory of t–norm–based rough inclusions.

Now, we would like to review some applications to rough mereological constructs.

3 Application 1: Granulation of Knowledge

As said above, indiscernibility classes of IND are regarded as elementary granules of knowledge, and their unions form a Boolean algebra of granules of knowledge relative to a given information system (U, A). Rough sets know also some other forms of granules, based on, e.g., entropy (see the paper by Ślęzak in [17].

Using a rough inclusion μ, or ν, one can produce granules on which a more subtle topological structure can be imposed. The tool is the class operator. Given r, and $u \in U$, we define a property $P^u_\mu(v, r)$ that holds iff $\mu(v, u, r)$, and then we form the class of this property: $g^\mu_r(u){=}Cls(P^u_\mu(v, r)$. Granules have some regular properties:

$$\begin{array}{c} 1.\ \text{if } y\ el\ u \text{ then } y\ el\ g^\mu_r(u) \\ 2.\ \text{if } v\ el\ g^\mu_r(u) \text{ and } w\ el\ v \text{ then } w\ el\ g^\mu_r(u) \\ 3.\ \text{if } \mu(v, u, r) \text{ then } v\ el\ g^\mu_r(u). \end{array} \tag{2}$$

Properties 1-3 follow from properties in sect. 2.2 and the fact that el is a partial order, in particular it is transitive.

The case of an archimedean rough inclusion. In case of a rough inclusion μ_t induced by an archimedean t–norm t, one may give a better description of granule behavior, stating the property 3 in (2) in a more precise way,

$$v\ el\ g^{\mu_t}_r(u) \text{ iff } \mu_t(v, u, r). \tag{3}$$

Rough inclusions on granules. Regarding granules as objects, calls for a procedure for evaluating rough inclusion degrees among granules. First, we have to define the notion of an element among granules. We let, for granules g, h,

$$\begin{array}{c} g\ el\ h \text{ iff } [z\ el\ g \text{ implies there is } t \text{ such that} \\ z\ el\ t, t\ el\ h], \end{array} \tag{4}$$

and, more generally, for granules g, h, and a rough inclusion μ,

$$\begin{array}{c} \mu(g, h, r) \text{ if and only if for } w\ el\ g \text{ there is } v \text{ such that} \\ \mu(w, v, r),\ v\ el\ h. \end{array} \tag{5}$$

Then: *μ is a rough inclusion on granules.* This procedure may be iterated to granules of granules, etc., etc. Let us note that due to our use of class operator (being, for our set theoretical representation of granules, the union of sets operator), we always remain on the level of collections of objects despite forming higher–level granules.

We also have,

$$\text{if } vingr\ g^{\mu_t}_r(u) \text{ then } g^{\mu_t}_s(v)\ ingr\ g^{\mu_t}_{t(r,s)}(u), \tag{6}$$

showing a kind of weak topology on granules.

Granular information systems. Given a rough inclusion μ on the set U of objects, we define an r*–net*, where $r \in (0,1)$, as a set $N_r = \{u_1, .., u_k\} \subset U$ such that the granule set $G_r = \{g_r^\mu(u_1), .., g_r^\mu(u_k)\}$ is a covering of U. For each of granules $g_r^\mu(u_j)$, $j \in \{1, .., k\}$, we select the decision value and values of conditional attributes in the set A by means of some strategies, respectively, $\mathcal{A}, \mathcal{D}$. The resulting decision system $(G_r, \mathcal{A}(A), \mathcal{D}(d))$ is the $(G_r, \mathcal{D}, \mathcal{A})$*–granular decision system*. Decision rules induced from the granular decision system can be regarded as an approximation to decision rules from the original system; one may expect the former will be shorter subrules of the latter in general.

Example 1. A simple example that illustrates the idea is given. Table 1 is a simple decision system ([17], p.18).

Table 1. A simple test table

obj	*a*1	*a*2	*a*3	*a*4	*d*
*o*1	1	1	1	2	1
*o*2	1	0	1	0	0
*o*3	2	0	1	1	0
*o*4	3	2	1	0	1
*o*5	3	1	1	0	0
*o*6	3	2	1	2	1
*o*7	1	2	0	1	1
*o*8	2	0	0	2	0

This system produces 14 decision rules generated by the RSES 2 system [19]:
(a1=1),(a2=1)⇒(d=1[1]) 1; (a1=1),(a2=0)⇒(d=0[1]) 1;
(a1=2),(a2=0)⇒(d=0[2]) 2; (a1=3),(a2=2)⇒(d=1[2]) 2;
(a1=3),(a2=1)⇒(d=0[1]) 1; (a1=1),(a2=2)⇒(d=1[1]) 1;
(a2=1),(a4=2)⇒(d=1[1]) 1; (a2=0),(a4=0)⇒(d=0[1]) 1;
(a2=0),(a4=1)⇒(d=0[1]) 1; (a2=2),(a4=0)⇒(d=1[1]) 1;
(a2=1),(a4=0)⇒(d=0[1]) 1; (a2=2),(a4=2)⇒(d=1[1]) 1;
(a2=2),(a4=1)⇒(d=1[1]) 1; (a2=0),(a4=2)⇒(d=0[1]) 1.

Applying the t-norm t_L with $r = .5$ and using the strategy of majority voting with random resolution of ties, we produce the table Table 2 of the granular counterpart to Table 1 with four granules $g1 - g4$, centered at objects, resp., $o1, o2, o3, o7$.

For Table 2, there are 10 rules generated by the system RSES:
(ga1=1)⇒(gd=1[2]) 2; (ga1=3)⇒(gd=0[1]) 1;
(ga1=2)⇒(gd=0[1]) 1; (ga2=1)⇒(gd=1[1]) 1;
(ga2=0)⇒(gd=0[2]) 2; (ga2=2)⇒(gd=1[1]) 1;
(ga3=1),(ga4=2)⇒(gd=1[1]) 1; (ga3=1),(ga4=0)⇒(gd=0[1]) 1;
(ga3=1),(ga4=1)⇒(gd=0[1]) 1; (ga3=0),(ga4=1)⇒(gd=1[1]) 1.

We call a rule r_1 *subordinated* to rule r_2 if the set of descriptors $(a = v)$ in the antecedent of r_1 is a subset of the set of descriptors in the antecedent of

Table 2. A granular decision system for Table 1

granobj	*ga*1	*ga*2	*ga*3	*ga*4	*gd*
*g*1	1	1	1	2	1
*g*2	3	0	1	0	0
*g*3	2	0	1	1	0
*g*4	1	2	0	1	1

r_2 and decision values are identical in both rules. This means that r_1 is shorter but has the same predictive ability. Comparing the two sets of rules, one finds that 60 percent of rules for Table 2 are subordinated to rules for table 1. This means that the rules for Table 2 approximate the rules for the original Table 1 to degree of 0.6. In connection with this, we state

RESEARCH PROBLEM 2: verify experimentally the feasibility of this approach to real data of importance. This implies software solutions as well.

4 Application 2: Rough Mereological Logics

Rough inclusions can be used to define logics for rough sets; for a rough inclusion μ on subsets of the universe U of an information system (U, A), we define an intensional logic RML^{μ}. We assume a set P of unary open predicates given, from which formulas are formed by means of connectives C of implication and N of negation; the intension $I(\mu)$ assigns to a predicate $\phi \in P$ a mapping $I(\mu)(\phi) : E \to [0, 1]$, where E is the family of exact sets (or, granules) defined in (U, A). For each predicate p its meaning in the set U is given as $[[p]] = \{u \in U : p(u)\}$.

For an exact set G, the extension of ϕ at G is defined as $I(\mu)^{\vee}_{G}(\phi)=I(\mu)(\phi)(G)$ and it is interpreted as the value of truth (or, the state of truth) of ϕ at G.

We adopt the following interpretation of logical connectives N of negation and C of implication,

$[[Np]] = U \setminus [[p]], [[Cpq]] = (U \setminus [[p]]) \cup [[q]]$.

These assignments of meaning extend by recursion from predicates in P to formulas.

The value $I(\mu)^{\vee}_{G}(\phi)$ of the extension of ϕ at an exact set G is defined as follows,

$$I(\mu)^{\vee}_{G}(\phi) \geq r \Leftrightarrow \mu(G, [[\phi]], r). \tag{7}$$

We call a meaningful formula ϕ a *theorem with respect to* μ if and only if $I(\mu)^{\vee}_{G}(\phi) = 1$ for each $G \in E$.

The case of the Łukasiewicz t–norm. We give some facts concerning the rough inclusion μ_{t_L} induced by the Łukasiewicz t–norm t_L; in this case we have by results of sect.2.3 that,

$$I(\mu_{t_L})^{\vee}_{G}(\phi) \geq r \Leftrightarrow \frac{|G \cap [[\phi]]|}{|G|} \geq r. \tag{8}$$

In what follows, $I(\mu_{t_L})^{\vee}_{G}(\phi)$ is identified with the value of $\frac{|G\cap[[\phi]]|}{|G|}$.

One verifies that,

$$I(\mu_{t_L})^{\vee}_{G}(N\phi) = 1 - I(\mu_{t_L})^{\vee}_{G}(\phi)), \tag{9}$$

and,

$$I(\mu_{t_L})^{\vee}_{G}(C\phi\psi) \leq 1 - I(\mu_{t_L})^{\vee}_{G}(\phi) + I(\mu_{t_L})^{\vee}_{G}(\psi). \tag{10}$$

The formula on the right hand side of inequality (10) is of course the Łuka - siewicz implication of many–valued logic. We may say that in this case the logic $RML^{\mu_{t_L}}$ is a sub–Łukasiewicz many–valued logic, meaning in particular, that if a sentential form of the formula $\phi(x)$ is a theorem of $[0,1]$–valued Łukasiewicz logic then $\phi(x)$ is a theorem of the logic RML.

One verifies directly that derivation rules:

(MP) $\frac{p(x), Cp(x)q(x)}{q(x)}$ (modus ponens)

and

(MT) $\frac{\neg q(x), Cp(x)q(x)}{\neg p(x)}$ (modus tollens)

are valid in the logic RML^{μ} for each regular rough inclusion μ. In the context of intensional logic RML, we may discuss modalities L (of necessity) and M (of possibility).

Necessity, possibility. We define, with the help of a regular rough inclusion μ, functors L of necessity and M of possibility (the formula $L\phi$ is read "it is necessary that ϕ" and the formula $M\phi$ is read: "it is possible that ϕ") with partial states of truth as follows,

$$I(\mu)^{\vee}_{G}(L\phi) \geq r \Leftrightarrow \mu(G, \underline{[[p(x)]]}, r), \tag{11}$$

and, similarly,

$$I(\mu)^{\vee}_{G}(\phi)) \geq r \Leftrightarrow \mu(G, \overline{[[p(x)]]}, r). \tag{12}$$

It seems especially interesting to look at operators L, M with respect to the rough inclusion μ_{t_L} of Łukasiewicz. Then,

In the logic $RML^{\mu_{t_L}}$, a meaningful formula $\phi(x)$ is satisfied necessarily (i.e., it is necessary in degree 1) with respect to an exact set G if and only if $G \subseteq \underline{[[\phi(x)]]}$; similarly, $\phi(x)$ is possible (i.e., possible in degree 1) with respect to the set G if and only if $G \subseteq \overline{[[\phi(x)]]}$.

Clearly, by duality of rough set approximations, the crucial relation,

$$I(\mu_{t_L})^{\vee}_{G}(L\phi) = 1 - I(\mu_{t_L})^{\vee}_{G}(MN\phi), \tag{13}$$

holds between the two modalities with respect to each rough inclusion μ.

A Calculus of modalities. We now may present within our intensional logic $RML^{\mu_{t_L}}$ an otherwise well–known fact, obtained within different frameworks by a few authors (e.g, Orłowska, Pawlak–Orłowska, Rasiowa–Skowron, Vakarelov, see [7]) that rough sets support modal logic S5.

Proposition 1. *The following formulas of modal logic are theorems of RML with respect to every regular rough inclusion* μ*:*

1. *(K)* $CL(Cp(x)q(x))CLp(x)Lq(x)$.
2. *(T)* $CLp(x)p(x)$.
3. *(S4)* $CLp(x)LLp(x)$.
4. *(S5)* $CMp(x)LMp(x)$.

RESEARCH PROBLEM 3: establish properties of rough mereological logics, in particular relations to fuzzy logics.

4.1 A Formalization of Calculus of Perceptions

An example of a flexibility and power of our calculus based on rough inclusions, is a formalization of calculus of perceptions, a phrase coined by L. Zadeh. Perceptions are vague statements often in natural language, and we interpret them semantically as fuzzy entities in the sense of fuzzy set theory of Zadeh. Fuzzy entities in turn form a hierarchy of predicates interpreted in the universe of an information system. A query related to the perception induces constraints interpreted as exact sets (granules); measuring the truth value of predicates constituting the formal rendering of a perception against those exact sets gives the truth value of perceptions.

Example 2. A very simple example illustrates the idea.

Premises: *Joan has a child of about ten years old.*

Query: *How old is Joan*?

We address this query with reference to knowledge encoded in Table 3, where *child* is the child age, and *age* is the mother age. We will use the t–norm t_L

Table 3. A decision system child age-mother age

object	*child*	age
1	15	58
2	10	42
3	10	30
4	24	56
5	28	62
6	40	67
7	25	60
8	26	63
9	38	70
10	16	38

The interpretation of the concept μ_{10} - "about ten", over the domain $D_{10} = [0, 30]$, is given as,

$$\mu_{120}(x) = \begin{cases} \frac{x}{5} \text{ for } x \in [0,5] \\ 1 \text{ for } x \in [5,15] \\ 2 - \frac{x}{15} \text{ for } x \in [15,30] \end{cases}$$

The interpretation of the concept "Old", over the domain $D_{Old} = [30, 70]$, is given as,

$$\mu_{Old}(x) = \begin{cases} 0.02(x-30) \text{ for } x \in [30,60] \\ 0.04(x-60) + 0.6 \text{ for } x \in [60,70] \end{cases} \quad (14)$$

The answer to the query will be presented as a fuzzy entity, defined as follows: given cut levels $a, b \in (0, 1)$ for notions "about ten", "Old", respectively; choice of a sets constraint on objects in Table 3, interpreted as a granule G, and then, choice of cut level b produces a meaning [age $\geq b$] for predicate age $\geq b$ induced from Table 3. For values of a, b, the value of $I(\mu)^{\vee}_{G}(\text{age} \geq b)$ is the truth degree of the statement: *"for given a, b, the age of Joan is at least the value at the cut level b with the truth degree of* $I(\mu)^{\vee}_{G}(\text{age} \geq b)$".

In our case, let $a = .5 = b$; then, the granule G defined by the interval,

$$\text{about ten}_{.5} = [2.5, 22.5], \quad (15)$$

is $G=\{1, 2, 3, 10\}$. Now, for $b = .5$, the meaning [age $\geq .5$] is $\{1, 4, 5, 6, 7, 8, 9\}$. The age defined by $b = .5$ is 55.

The truth degree of the statement:

" the age of Joan is at least 55" is $\frac{|\{1,2,3,10\} \cap \{1,4,5,6,7,9\}|}{|\{1,2,3,10\}|}$=.25, for the given a, b. The complete answer is thus a fuzzy set over the domain $[0, 1]^2 \times D_{age}$.

RESEARCH PROBLEM 4: construct an interface for inducing constraints and fuzzy predicates from a vague input in Natural Language (a restricted formalized subset of).

5 Application 3: Networks of Cognitive Agents

A granular agent ag in its simplest form is a tuple

$$ag^* = (U_{ag}, A_{ag}, \mu_{ag}, Pred_{ag}, UncProp_{ag}, GSynt_{ag}, LSynt_{ag}),$$

where $(U_{ag}, A_{ag}) = is_{ag}$ is an information system of the agent ag, μ_{ag} is a rough inclusion induced from is_{ag}, and $Pred_{ag}$ is a set of first–order predicates interpreted in U_{ag} in the way indicated in Sect. IV. $UncProp_{ag}$ is the function that describes how uncertainty measured by rough inclusions at agents connected to ag propagates to ag. The operator $GSynt_{ag}$, the granular synthesizer at ag, takes granules sent to the agent from agents connected to it, and makes those granules into a granule at ag; similarly $LSynt_{ag}$, the logic synthesizer at ag, takes formulas sent to the agent ag by its connecting neighbors and makes them into a formula describing objects at ag.

A network of granular agents is a directed acyclic graph $N = (Ag, C)$, where Ag is its set of vertices, i.e., granular agents, and C is the set of edges, i.e., connections among agents, along with disjoint subsets $In, Out \subset Ag$ of, respectively, input and output agents.

5.1 On Workings of an Elementary Subnetwork of Agents

We consider an agent $ag \in Ag$ and - for simplicity reasons - we assume that ag has two incoming connections from agents ag_1, ag_2; the number of outgoing connections is of no importance as ag sends along each of them the same information.

We assume that each agent is applying the rough inclusion μ_{t_L} induced by the Łukasiewicz t–norm t_L, see sect. 2.3, in its granulation procedure; also, each agent is applying the rough inclusion on sets of the form given in sect. 2.3 in evaluations related to extensions of formulae intensions.

Example 3. The parallel composition of information systems. Clearly, there exists a fusion operator o_{ag} that assembles from objects $x \in U_{ag_1}, y \in U_{ag_2}$ the object $o(x, y) \in U_{ag}$; we assume that $o_{ag} = id_{ag_1} \times id_{ag_2}$, i.e., $o_{ag}(x, y) = (x, y)$. Similarly, we assume that the set of attributes at ag, equals: $A_{ag} = A_{ag_1} \times A_{ag_2}$, i.e., attributes in A_{ag} are pairs (a_1, a_2) with $a_i \in A_{ag_i}$ $(i = 1, 2)$ and that the value of this attribute is defined as: $(a_1, a_2)(x, y)=(a_1(x), a_2(y))$.

It follows that the condition holds:

$o_{ag}(x, y) IND_{ag} o_{ag}(x', y')$ iff $x IND_{ag_1} x'$ and $y IND_{ag_2} y'$.

Concerning the function $UncProp_{ag}$, we consider objects x, x', y, y'; clearly,

$$DIS_{ag}(o_{ag}(x, y), o_{ag}(x', y')) \subseteq DIS_{ag_1}(x, x') \times A_{ag_2} \cup A_{ag_1} \times DIS_{ag_2}(y, y'), \quad (16)$$

and hence,

$$|DIS_{ag}(o_{ag}(x, y), o_{ag}(x', y'))| \leq |DIS_{ag_1}(x, x')| \cdot |A_{ag_2}| + |A_{ag_1}| \cdot |DIS_{ag_2}(y, y')|. \quad (17)$$

By (17),

$$\begin{array}{c} \mu_{ag}(o_{ag}(x, y), o_{ag}(x', y'), t) \\ = 1 - \frac{|DIS_{ag}(o_{ag}(x,y), o_{ag}(x',y'))|}{|A_{ag_1}| \cdot |A_{ag_2}|} \\ \geq 1 - \frac{|DIS_{ag_1}(x,x')| \cdot |A_{ag_2}| + |A_{ag_1}| \cdot |DIS_{ag_2}(y,y')|}{|A_{ag_1}| \cdot |A_{ag_2}|} \\ = 1 - \frac{|DIS_{ag_1}(x,x')|}{|A_{ag_1}|} + 1 - \frac{|DIS_{ag_2}(y,y')|}{|A_{ag_2}|} - 1. \end{array} \quad (18)$$

It follows that,

$$\text{if } \mu_{ag_1}(x, x', r), \mu_{ag_2}(y, y', s) \text{ then} \mu_{ag}(o_{ag}(x, y), o_{ag}(x', y'), t_L(r, s)). \quad (19)$$

Hence, $UncProp(r, s) = t_L(r, s)$, the value of the Łukasiewicz t–norm t_L on the pair (r, s).

In consequence, the granule synthesizer $GSynt_{ag}$ can be defined in our example as,

$$GSynt_{ag}(g_{ag_1}(x, r), g_{ag_2}(y, s)) = (g_{ag}(o_{ag}(x, y), t_L(r, s)). \quad (20)$$

The definition of logic synthesizer $LSynt_{ag}$ follows directly from our assumptions,

$$LSynt_{ag}(\phi_1, \phi_2) = \phi_1 \wedge \phi_2. \tag{21}$$

Finally, we consider extensions of our logical operators of intensional logic. We have for the extension $I(\mu_{ag})^{\vee}_{GSynt_{ag}(g_1,g_2)}(LSynt_{ag}(\phi_1, \phi_2))$:

$$I(\mu_{ag})^{\vee}_{GSynt_{ag}(g_1,g_2)}(LSynt_{ag}(\phi_1, \phi_2)) = I(\mu_{ag_1})^{\vee}_{g_1}(\phi_1) \cdot I(\mu_{ag_2})^{\vee}_{g_2}(\phi_2), \tag{22}$$

which follows directly from (20), (21).

Thus, in our example, each agent works according to regular t–norms: the Łukasiewicz t–norm on the level of rough inclusions and uncertainty propagation and the Menger (product) t–norm $\cdot$ on the level of extensions of logical intensions.

RESEARCH PROBLEM 5: explore other models of knowledge fusion introducing synergy effects.

6 Conclusion and Acknowledgements

We have presented basics of rough mereological approach along with some selected applications to granular computing, perception calculus, as well as problems whose solutions would in our opinion advance rough set theory. We are grateful to many colleagues for cooperation in many ways and particularly to Professors Guoyin Wang and Qing Liu for their kind invitation to China. The referees are thanked for comments. Clearly, the author is responsible for all errors.

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