



## The Fuzzy Logic Concept

The way that people think is inherently fuzzy. The way that we perceive the world is continually changing and cannot always be defined in true or false statements. Take for example the set of all the apples and all the apple cores in the world. Now take one of those apples; it belongs to the set of all apples. Now take a bite out of that apple; it is still an apple right? If so, it still belongs to the set of apples. After several more bites have been taken and you are left with an apple core and it belongs to the set of apple cores. At what point did the apple cross over from being an apple to being an apple core? What if you could get one more bite out of that apple core, does that move it into a different set?

The definition of the apple and apple core sets are too strictly defined when looking at the process of eating an apple. The area between the two sets is not clearly defined since the object cannot belong to the set of apples and apple cores because, by definition, an apple core is NOT an apple. The sets defining apples and apple cores need to be redefined as fuzzy sets.

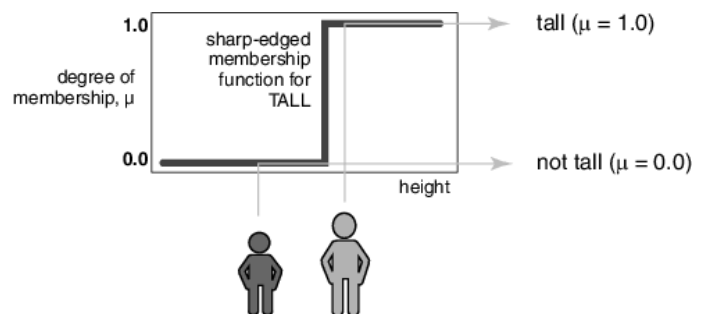


A fuzzy set allows for its members to have degrees of membership. If the value of 1 is assigned to objects entirely within the set and a 0 is assigned to objects outside of the set, then any object partially in the set will have a value between 0 and 1. The number assigned to the object is called its degree of membership in the set. So an apple with one bite out of it may have a degree of membership of 0.9 in the set of apples. This does not mean that it has to have a degree of membership of 0.1 in the set of apple cores though. However as the apple is eaten it loses its membership in the fuzzy set of apples and gains membership in the fuzzy set of apple cores.

### Defining Fuzzy Sets

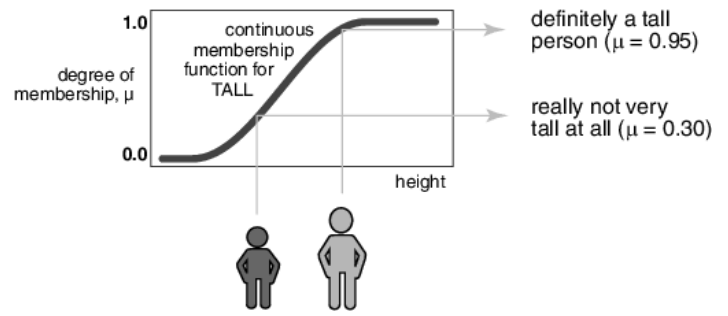
In mathematics a set, by definition, is a collection of things that belong to some definition. Any item either belongs to that set or does not belong to that set. Let us look at another example; the set of tall men. We shall say that people taller than or equal to 6 feet are tall. This set can be represented graphically as follows:

The function shown above describes the membership of the 'tall' set, you are either in it or you are not in it. This sharp edged membership function works nicely for binary operations and mathematics, but it does not work as nicely in describing the real world. The membership function makes no distinction between somebody who is 6'1" and someone who is 7'1", they are both simply tall. Clearly there is a significant difference between the two heights. The other side of this lack of distinction is the difference between a 5'11" and 6' man. This is only a difference of one inch; however this membership function just says one is tall and the other is not tall.



The fuzzy set approach to the set of tall men provides a much better representation of the tallness of a person. The set, shown below, is defined by a continuously inclining function.

The membership function defines the fuzzy set for the possible values underneath of it on the horizontal axis. The vertical axis, on a scale of 0 to 1, provides the membership value of the height in the fuzzy set. So for the two people shown above the first person has a membership of 0.3 and so is not very tall. The second person has a membership of 0.95 and so he is definitely tall. He does not, however, belong to the set of tall men in the way that bivalent sets work; he has a high degree of membership in the fuzzy set of tall men.



### Defining Fuzzy Sets Mathematically

Fuzzy sets were first proposed by Lofti A. Zadeh in his 1965 paper entitled none other than: *Fuzzy Sets*. This paper laid the foundation for all fuzzy logic that followed by mathematically defining fuzzy sets and their properties. The definition of a fuzzy set then, from Zadeh's paper is:

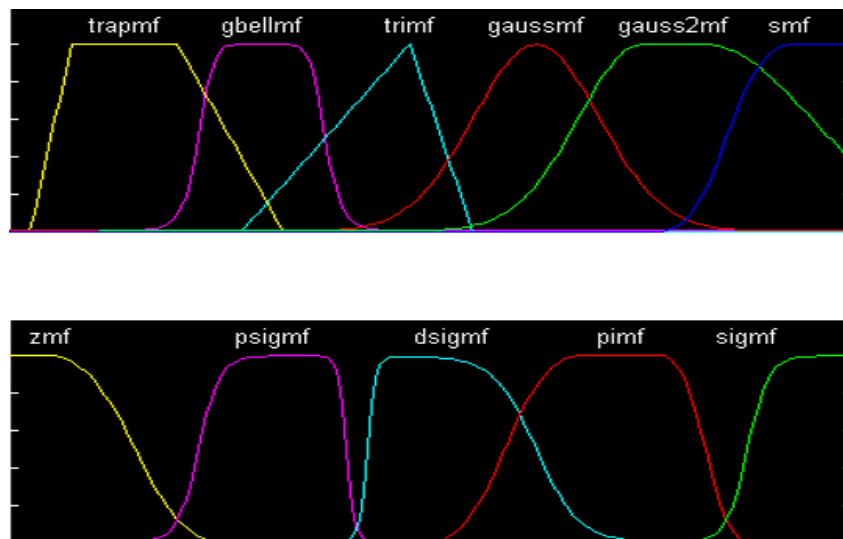
Let  $X$  be a space of points, with a generic element of  $X$  denoted by  $x$ . Thus  $X = \{x\}$ .

A fuzzy set  $A$  in  $X$  is characterized by a membership function  $f_A(x)$  which associates with each point in  $X$  a real number in the interval  $[0,1]$ , with the values of  $f_A(x)$  at  $x$  representing the "grade of membership" of  $x$  in  $A$ . Thus, the nearer the value of  $f_A(x)$  to unity, the higher the grade of membership of  $x$  in  $A$ .

*from Fuzzy Sets, by Lofti A. Zadeh*

This definition of a fuzzy set is like a superset of the definition of a set in the ordinary sense of the term. The grades of membership of 0 and 1 correspond to the two possibilities of truth and false in an ordinary set. The ordinary Boolean operators that are used to combine sets will no longer apply; we know that 1 AND 1 is 1, but what is 0.7 AND 0.3? This will be covered in the fuzzy operations section.

Membership functions for fuzzy sets can be defined in any number of ways as long as they follow the rules of the definition of a fuzzy set. The Shape of the membership function used defines the fuzzy set and so the decision on which type to use is dependent on the purpose. The membership function choice is the subjective aspect of fuzzy logic; it allows the desired values to be interpreted appropriately. The most common membership functions are shown below:



## Defining Fuzzy Operations

Traditional Bivalent logic uses the Boolean operators AND, OR, and NOT to perform the intersect, union and complement operations. These operators work well for bivalent sets and can be essentially defined using the following truth table.

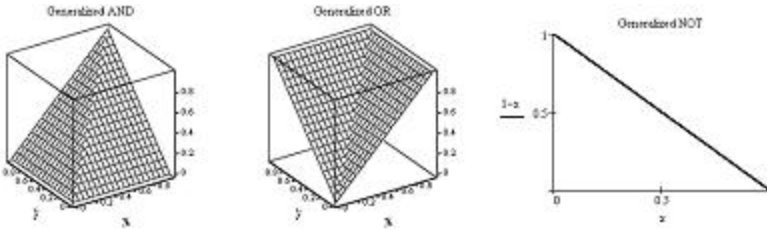
x	y	x AND y	x OR y	NOT x
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	0

The truth table above works fine for bivalent logic but fuzzy logic does not have a finite set of possibilities for each input; this makes for an infinitely large truth table. The operators need to be defined as functions for all possible fuzzy values, that is, all real numbers from 0 to 1 inclusive. Fuzzy logic is actually a superset of bivalent logic since it includes the bivalent options (0, 1) as well as all reals in between, so a generalized form of these operators will be useful. The generalized form for these three operators is:

x AND y	$\min(x,y)$
x OR y	$\max(x,y)$
NOT x	$1 - x$

Using these definitions they can be applied to all of the bivalent combinations above as well as some fuzzy number combinations. The truth table for this can be seen below:

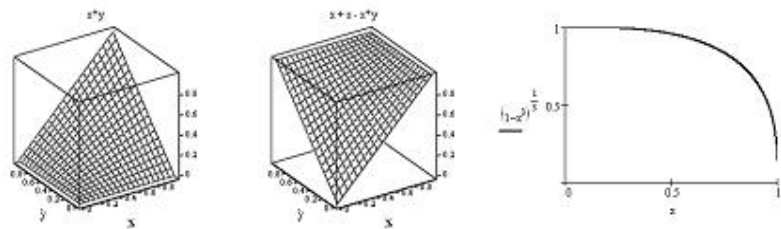
x	y	$\min(x,y)$	$\max(x,y)$	$1 - x$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	0
0.2	0.5	0.2	0.5	0.8
0.7	0.2	0.2	0.7	0.3
0.6	0.6	0.6	0.6	0.4



These generalized definitions of the operators work well for fuzzy numbers as well as bivalent sets. The behavior of the new generalized definitions of AND, OR and NOT can be visualized below for all possible inputs of  $x$  and  $y$ :

The surfaces above show the behavior of the generalized AND, OR and NOT operators. There are many other possible definitions for the different types of operators but they all share similar properties. Mathematical definitions of the AND operator are called triangular norms or t-norms, this name is derived from the shape of the generalized AND. A t-norm is, by definition, a binary operator with both operand and the result in  $[0, 1]$ , is commutative, associative, has 1 as an identity, and is increasing in each variable. Mathematical definitions of the OR operator have all the same properties of t-norms except that they have 0 as an identity; they are called t-conorms. The NOT operator can be redefined as long as it is a continuous, strictly decreasing function within  $[0, 1]$ . Some examples of different t-norms, t-conorms and negations can be seen below:

These are only one example of each different type of operator; many others for each operator exist. The choice of which definition to use will affect the way fuzzy sets are combined and this should be kept in mind when selecting an operators defining function. When an operator definition is selected it should be used consistently so that set combinations remain consistent.



### Making Fuzzy Decisions

Most decisions that people make are logical decisions, they look at the situation and make a decision based on the situation. The generalized form of such a decision is called a generalized modus ponens, which is in the form:

**If P, then Q.**  
**P.**  
**Therefore, Q.**

This form of logical reasoning is fairly strict, Q can only be if P. Fuzzy logic loosens this strictness by saying that Q can mostly be if P is mostly or:

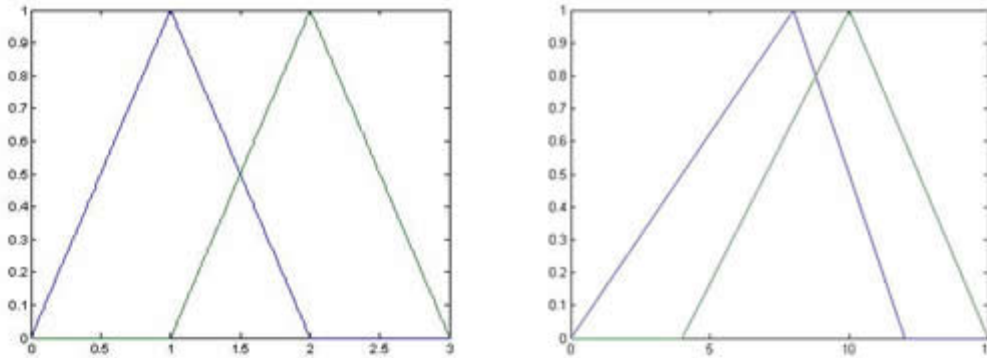
**If P, then Q.**  
**mostly P.**  
**Therefore, mostly Q.**

Where P and Q are now fuzzy numbers. The reasoning above requires a set of rules to be defined. These rules are linguistic rules to relate different fuzzy sets and numbers. The general form of these rules is: "if  $x$  is A then  $y$  is B," where  $x$  and  $y$  are fuzzy numbers in the fuzzy sets A and B respectively. These fuzzy sets are defined by membership functions. There can be any number of input and output membership functions for the same input as well, depending on the number of rules in the system. For example, a system could have membership functions that represent slow, medium, and fast as inputs.

The linguistic rules are used to define the relation between the input and the output, but how exactly are the output fuzzy values determined? There are several ways to determine the answer based on the inputs, mainly the Mamdani, Larsen, Takagi-Sugeno-Kang, and Tsukamoto inference and aggregation methods. Firstly, we must describe the basic general set of rules, they will be a set of rules that have one input in a fuzzy set and one output in a fuzzy set:

If  $x$  is  $A_i$  then  $y$  is  $B_i, i=1,2,\dots,n$

Let us look at a system that has two input membership functions ( $A_1, A_2$ ) and two output membership functions ( $B_1, B_2$ ). These membership functions, shown below, define the fuzzy sets  $A$  and  $B$  in the above general inference rule.



$A_1$  and  $A_2$  are shown on the left, with  $A_1$  in blue and  $A_2$  in green. On the right  $B_1$  is blue and  $B_2$  is green. We will be using the Mamdani inference model to combine the sets and rules. The Mamdani inference model is:

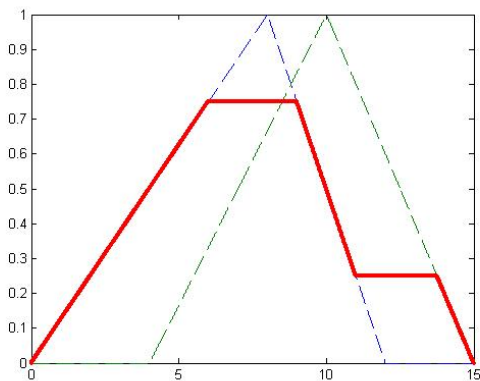
$$R(x,y) = \mu_{A_1}(x) \wedge \mu_{A_2}(x) \wedge \mu_{B_1}(y) \vee \mu_{A_2}(x) \wedge \mu_{A_1}(x) \wedge \mu_{B_2}(y)$$

Using this model will give an aggregate fuzzy set,  $R$  that uses the input values in  $A_1$  and  $A_2$  to modify and combine  $B_1$  and  $B_2$ . The input membership functions, as well as the output membership functions, are overlapping; this means that an input value can have membership in both membership functions, or in only one. If the input value has membership in a function, than any rule using that membership function is said to 'fire' and produce a result. These results are then aggregated using the Mamdani model, or a different model.

Let us then pick an input value that has membership function in  $A_1$  and  $A_2$ , 1.25, this will cause both rules to fire. The value 1.25 has a membership of 0.75 in  $A_1$  and a membership of 0.25 in  $A_2$ . Using the Mamdani model and these inputs the resulting aggregate output will be:

$$[A_1(1.25) \wedge B_1(y)] \vee [A_2(1.25) \wedge B_2(y)]$$

When all of these combinations have been made, the aggregate output membership function (red), as well as  $B_1$  and  $B_2$  (dashed) are shown below:



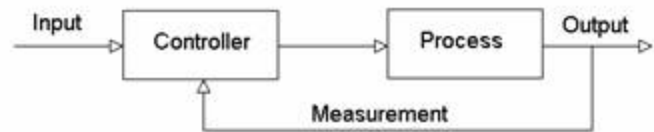
This aggregate fuzzy membership function is the result of the rule based inference decision making process. To get a finite number as an output we need to go through the defuzzification process. Defuzzification is a method that produces a number that best represents, and consistently represents the fuzzy set. There are many ways to do this with most of them being some type of averaging method. The most common is the centroid method, this calculates the center of area of the fuzzy set and uses the value at which this occurs as the defuzzified output. Other methods include the bisector, largest of maximum, smallest of maximum, and middle of maximum. For the above aggregate fuzzy set, the different defuzzification methods produce these finite values shown below. So, if the most common

method, centroid, is used, the finite result would be 7.319.

Defuzz Method	Result
Centroid	7.319
Bisector	7.230
Largest of Max	9
Smallest of Max	6
Middle of Max	7.5

## Fuzzy Engineering

There is more to fuzzy logic than some interesting math, it has some impressive applications in engineering. The main application of fuzzy logic in engineering is in the area of control systems. The definition of a control system, given by Richard Dorf in *Modern Control Systems* is: "An interconnection of components forming a system configuration that will provide a desired response." This means that a control system needs to know the desired response (input) and it needs to process this input and attempt to achieve it. The general control system can then be summarized with the following diagram:



The process is the system that is being controlled and cannot typically be changed. The controller then, must take the input and also take measurements from the process and use this information to generate the appropriate input to the process. A basic example of a controller would be a summing point that will provide the difference between input and output to the process, whereas a more advanced controller would be a PID controller. A fuzzy logic based controller will use fuzzy membership functions and inference rules to determine the appropriate process input. Designing a fuzzy controller is a more intuitive approach to controller design since it uses a comprehensible linguistic rule base.

A fuzzy controller can be broken down into three main processes. The first of these is the fuzzification, this uses defined membership functions to process the inputs and to fuzzify them. These fuzzified inputs are then used in the second part, the rule-based inference system. This system uses previously defined linguistic rules to generate a fuzzy response. The fuzzy response is then defuzzified in the final process: defuzzification. This process will provide a real number as an output.

Designing a fuzzy controller can be done with several different computer based tools, the tool we will be using is the Fuzzy Logic Toolbox in MATLAB with Simulink. This toolbox provides a GUI for defining membership functions and inference rules and can be integrated with Simulink. The tutorials below will walk you through the process of designing a fuzzy controller using the Fuzzy Logic Toolbox. These tutorials assume that you are familiar with MATLAB and Simulink. If not, go to <http://ctms.engin.umich.edu/CTMS/index.php?aux=Home> for a MATLAB Tutorial or go <http://ece.wpi.edu/courses/es301/sim/simulink.html> for a Simulink Tutorial.

Select the desired Tutorial. The first tutorial shows you how to get create a fuzzy controller and how a fuzzy controller behaves. The second advanced Tutorial applies a fuzzy controller to a process.

## Fuzzy Logic Applications

Almost any control system can be replaced with a fuzzy logic based control system. This may be overkill in many places however it simplifies the design of many more complicated cases. So fuzzy logic is not the answer to everything, it must be used when appropriate to provide better control. If a simple closed loop or PID controller works fine then there is no need for a fuzzy controller. There are many cases when tuning a PID controller or designing a control system for a complicated system is overwhelming, this is where fuzzy logic gets its chance to shine.

One of the most famous applications of fuzzy logic is that of the Sendai Subway system in Sendai, Japan. This control of the Nanboku line, developed by Hitachi, used a fuzzy controller to run the train all day long. This made the line one of the smoothest running subway systems in the world and increased efficiency as well as stopping time. This is also an example of the earlier acceptance of fuzzy logic in the east since the subway went into operation in 1988. For more information on this see: <http://sipi.usc.edu/~kosko/Scientific%20American.pdf>(pdf) or [http://www.smart.sunderland.ac.uk/f\\_succ.htm](http://www.smart.sunderland.ac.uk/f_succ.htm)

The most tangible applications of fuzzy logic control have appeared commercial appliances. Specifically, but not limited to heating ventilation and air conditioning (HVAC) systems. These systems use fuzzy logic thermostats to control the heating and cooling, this saves energy by making the system more efficient. It also keeps the temperature more steady than a traditional thermostat. For more information on this application see: [http://www.fuzzytech.com/e/e\\_a\\_ea.html](http://www.fuzzytech.com/e/e_a_ea.html)

Another significant area of application of fuzzy control is in industrial automation. Fuzzy logic based PLCs have been developed by companies like [Moeller](#). These PLCs, as well as other implementations of fuzzy logic, can be used to control any number of industrial processes. For some examples see: [http://www.fuzzytech.com/e/e\\_a\\_plc.html](http://www.fuzzytech.com/e/e_a_plc.html)

Fuzzy logic also finds applications in many other systems. For example, the [MASSIVE](#) 3D animation system for generating crowds uses fuzzy logic for artificial intelligence. This program was used extensively in the making of the Lord of the Rings trilogy as well as The Lion, The Witch and the Wardrobe films.

As a final example of fuzzy logic, it can be used in areas other than simply control. Fuzzy logic can be used in any decision making process such as signal processing or data analysis. An example of this is a fuzzy logic system that analyzes a power system and diagnoses any harmonic disturbance issues. The system analyzes the fundamental voltage, as well as third, fifth and seventh harmonics as well as the temperature to determine if there is cause for concern in the operation of the system. A complete explanation of this project can be found in this paper: [Harmonic Distortion Diagnostic using Fuzzy Logic](#)(pdf)

## History and Objections

Fuzzy logic was introduced in 1965 by Lofti A Zadeh in his paper "Fuzzy Sets". Zadeh and others continued to develop fuzzy logic at that time. The idea of fuzzy sets and fuzzy logic were not accepted well within academic circles because some of the underlying mathematics had not yet been explored. The applications of fuzzy logic were slow to develop because of this, except in the east. In Japan specifically fuzzy logic was fully accepted and implemented in products simply because fuzzy logic worked, regardless of whether mathematicians agreed or not. The success of many fuzzy logic based products in Japan in the early 80s led to a revival in fuzzy logic in the US in the late 80s. Since that time America has been playing catch up with the east in the area of fuzzy logic.

Some of the objections that faced fuzzy logic in its early days are shown below. Note that Professor William Kahan is Lofti Zadeh colleague at UC Berkeley. Most objections to fuzzy logic have since faded due to the success of fuzzy applications.

"Fuzzy theory is wrong, wrong, and pernicious. What we need is more logical thinking, not less. The danger of fuzzy logic is that it will encourage the sort of imprecise thinking that has brought us so much trouble. [Fuzzy logic is the cocaine of science.](#)"

-Professor William Kahan UC Berkeley

"'Fuzzification' is a kind of scientific permissiveness. It tends to result in socially appealing slogans unaccompanied by the discipline of hard scientific work and patient observation."

-Professor Rudolf Kalman UFlorida

"Fuzziness is probability in disguise. I can design a controller with probability that could do the same thing that you could do with fuzzy logic."

-Professor Myron Tribus, on hearing of the fuzzy-logic control of the Sendai subway system IEEE Institute, may 1988.

## References and Links

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