

Four Important Number Systems

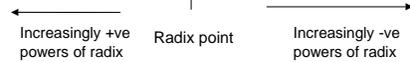
System	Why?	Remarks
Decimal	Base 10: (10 fingers)	Most used system
Binary	Base 2: On/Off systems	3-4 times more digits than decimal
Octal	Base 8: Shorthand notation for working with binary	3 times less digits than binary
Hex	Base 16	4 times less digits than binary



Positional Number Systems

- Have a radix r (base) associated with them.
- In the decimal system, $r = 10$:
 - Ten symbols: 0, 1, 2, ..., 8, and 9
 - More than 9 move to next position, so each position is power of 10
 - Nothing special about base 10 (used because we have 10 fingers)
- What does 642.391_{10} mean?

$$6 \times 10^2 + 4 \times 10^1 + 2 \times 10^0 + 3 \times 10^{-1} + 9 \times 10^{-2} + 1 \times 10^{-3}$$



Positional Number Systems(2)

- What does 642.391_{10} mean?

Base 10 (r)	10^2 (100)	10^1 (10)	10^0 (1)	10^{-1} (0.1)	10^{-2} (0.01)	10^{-3} (0.001)
Coefficient (a_i)	6	4	2	3	9	1
Product: $a_i \times r^i$	600	40	2	0.3	0.09	0.001
Value	= $600 + 40 + 2 + 0.3 + 0.09 + 0.001 = 642.391$					

- Multiply each digit by appropriate power of 10 and add them together

In general:
$$\sum_{i=-m}^n a_i \times r^i$$



Positional Number Systems(3)

Number system	Radix	Symbols
Binary	2	{0, 1}
Octal	8	{0, 1, 2, 3, 4, 5, 6, 7}
Decimal	10	{0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
Hexadecimal	16	{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f}



Binary Number System

Decimal	Binary	Decimal	Binary
0	0000	8	1000
1	0001	9	1001
2	0010	10	1010
3	0011	11	1011
4	0100	12	1100
5	0101	13	1101
6	0110	14	1110
7	0111	15	1111



Octal Number System

Decimal	Octal	Decimal	Octal
0	0	8	10
1	1	9	11
2	2	10	12
3	3	11	13
4	4	12	14
5	5	13	15
6	6	14	16
7	7	15	17



Hexadecimal Number System

Decimal	Hex	Decimal	Hex
0	0	8	8
1	1	9	9
2	2	10	A
3	3	11	B
4	4	12	C
5	5	13	D
6	6	14	E
7	7	15	F



Four Number Systems

Decimal	Binary	Octal	Hex	Decimal	Binary	Octal	Hex
0	0000	0	0	8	1000	10	8
1	0001	1	1	9	1001	11	9
2	0010	2	2	10	1010	12	A
3	0011	3	3	11	1011	13	B
4	0100	4	4	12	1100	14	C
5	0101	5	5	13	1101	15	D
6	0110	6	6	14	1110	16	E
7	0111	7	7	15	1111	17	F



Conversion: Binary to Decimal

Binary \longrightarrow Decimal

$1101.011_2 \longrightarrow (??)_{10}$

r	$2^3(8)$	$2^2(4)$	$2^1(2)$	$2^0(1)$	$2^{-1}(0.5)$	$2^{-2}(0.25)$	$2^{-3}(0.125)$
a_j	1	1	0	1	0	1	1
$a_j \cdot r$	8	4	0	1	0	0.25	0.125
$(1101.011)_2 = 8 + 4 + 1 + 0.25 + 0.125 = 13.375$							

$$1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} = 13.375_{10}$$

Binary point



Conversion: Decimal to Binary

A decimal number can be converted to binary by repeated division by 2 if it is an integer

number	$\div 2$	Remainder		
155	77	1	Least Significant Bit (LSB)	
77	38	1		
38	19	0	Arrange remainders in reverse order	
19	9	1		
9	4	1		
4	2	0		
2	1	0		
1	0	1		Most Significant Bit (MSB)

$\longrightarrow 155_{10} = 10011011_2$



Conversion: Decimal to Binary

- If the number includes a radix point, it is necessary to separate the number into an integer part and a fraction part, each part must be converted differently.

Decimal \longrightarrow Binary
 $(27.375)_{10} \longrightarrow (???)_2$

number	$\div 2$	Remainder	number	$\times 2$	Integer
27	13	1	0.375	0.75	0
13	6	1	0.75	1.50	1
6	3	0	0.50	1.0	1
3	1	1			
1	0	1			

Arrange in order: 011

Arrange remainders in reverse order: 11011

$\Rightarrow 27.375_{10} = 11011.011_2$



Conversion: Octal to Binary

Octal \longrightarrow Binary

$345.560_8 \longrightarrow (???)_2$

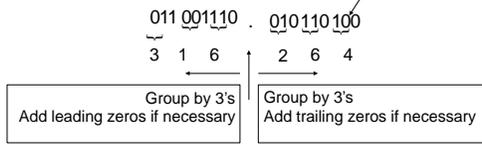
$\underbrace{3}_4 \underbrace{4}_4 \underbrace{5}_4 \underbrace{5}_4 \underbrace{6}_4 \underbrace{0}_4 \underbrace{2}_4$
 011 100 101 101 110 000 010

$345.560_8 = 11100101.10111000001_2$



Conversion: Binary to Octal

Binary \longrightarrow Octal
 $11001110.0101101_2 \longrightarrow (??)_8$ Note trailing zeros

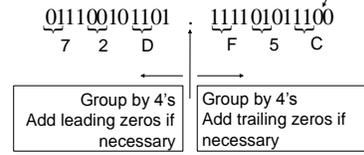


$$11001110.0101101_2 = 316.264_8$$



Conversion: Binary to Hex

Binary \longrightarrow Hex
 $11100101101.1111010111_2 \longrightarrow (??)_{16}$ Note trailing zeros

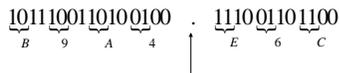


$$= 72D.F5C_{16}$$



Conversion: Hex to Binary

Hex \longrightarrow Binary
 $B9A4.E6C_{16} \longrightarrow (??)_2$



$$1011100110100100.1110011011_2$$



Conversion: Hex to Decimal

Hex \longrightarrow Decimal
 $B63.4C_{16} \longrightarrow (??)_{10}$

	16^2	16^1	16^0	16^{-1}	16^{-2}
B (=11)		6	3	4	C (=12)
	$= 2816 + 96 + 3 + 0.25 + 0.046875 = 2915.296875$				

$$11 \times 16^2 + 6 \times 16^1 + 3 \times 16^0 + 4 \times 16^{-1} + 12 \times 16^{-2} = 2915.296875_{10}$$



Conversion: Activity 1

- Convert the hexadecimal number A59.FCE to binary
- Convert the decimal number 166.34 into binary



Binary Numbers

- How many distinct numbers can be represented by n bits?

No. of bits	Distinct nos.
1	2 {0,1}
2	4 {00, 01, 10, 11}
3	8 {000, 001, 010, 011, 100, 101, 110, 111}
n	2^n

- Number of permutations double with every extra bit
- 2^n unique numbers can be represented by n bits



2's Complement Notation

Let N be an n bit number and $\tilde{N}(2)$ be the 2's Complement of the number. Then,

$$\tilde{N}(2) = 2^n - |N|$$

- Again, the idea is to leave positive numbers as is, but to represent negative numbers by the 2's C of their magnitude.
- Example: Let $n = 5$. What is 2's C representation for +11 and -13?
 - +11 is represented as 01011 (as usual in binary)
 - 13 is represented by 2's complement of its magnitude (13)



2's Complement Notation (2)

- 2's C representation can be computed in 2 ways:
 - Method 1: 2's C representation of -13 is $2^5 - |N| = (32 - 13)_{10} = (19)_{10} = (10011)_2$
 - Method 2: For -13, the magnitude is $13 = (01101)_2$
 - The 2's C representation is obtained by adding 1 to the 1's C of the magnitude
 - $2^5 - |N| = (2^5 - 1 - |N|) + 1 = 1's C + 1$

$$01101 \xrightarrow{1's C} 10010 \xrightarrow{add 1} 10011$$



Comparing All Signed Notations

4-bit No.	SM	1's C	2's C
0000	+0	+0	0
0001	1	1	1
0010	2	2	2
0011	3	3	3
0100	4	4	4
0101	5	5	5
0110	6	6	6
0111	7	7	7
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

- In all 3 representations, a -ve number has a 1 in MSB location
- To handle -ve numbers using n bits,
 - 2^{n-1} symbols can be used for positive numbers
 - 2^{n-1} symbols can be used for negative numbers
- In 2's C notation, only 1 combination used for 0



Unsigned Binary Integers

- Given an n -bit number

$$x = x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \dots + x_12^1 + x_02^0$$
- Range: 0 to $2^n - 1$
- Example
 - 0000 0000 0000 0000 0000 0000 1011₂

$$= 0 + \dots + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$

$$= 0 + \dots + 8 + 0 + 2 + 1 = 11_{10}$$
- Using 32 bits
 - 0 to +4,294,967,295

§ 2.4 Signed and Unsigned Numbers



2's-Complement Signed Integers

- Given an n -bit number

$$x = -x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \dots + x_12^1 + x_02^0$$
- Range: -2^{n-1} to $2^{n-1} - 1$
- Example
 - 1111 1111 1111 1111 1111 1111 1100₂

$$= -1 \times 2^{31} + 1 \times 2^{30} + \dots + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$$

$$= -2,147,483,648 + 2,147,483,644 = -4_{10}$$
- Using 32 bits
 - 2,147,483,648 to +2,147,483,647



2's-Complement Signed Integers(2)

- Bit 31 is sign bit
 - 1 for negative numbers
 - 0 for non-negative numbers
- Non-negative numbers have the same unsigned and 2's-complement representation
- Some specific numbers
 - 0: 0000 0000 ... 0000
 - 1: 1111 1111 ... 1111
 - Most-negative: 1000 0000 ... 0000
 - Most-positive: 0111 1111 ... 1111



Signed Negation

- Complement and add 1
 - Complement means $1 \rightarrow 0, 0 \rightarrow 1$

$$x + \bar{x} = 1111\dots111_2 = -1$$

$$\bar{\bar{x}} + 1 = -x$$

- Example: negate +2
 - $+2 = 0000\ 0000 \dots 0010_2$
 - $-2 = 1111\ 1111 \dots 1101_2 + 1$
 $= 1111\ 1111 \dots 1110_2$



Sign Extension

- Representing a number using more bits
 - Preserve the numeric value
- In MIPS instruction set
 - `addi`: extend immediate value
 - `lb`, `lh`: extend loaded byte/halfword
 - `beq`, `bne`: extend the displacement
- Replicate the sign bit to the left
 - c.f. unsigned values: extend with 0s
- Examples: 8-bit to 16-bit
 - $+2: 0000\ 0010 \Rightarrow 0000\ 0000\ 0000\ 0010$
 - $-2: 1111\ 1110 \Rightarrow 1111\ 1111\ 1111\ 1110$

