Chapter 5

Digital Filter

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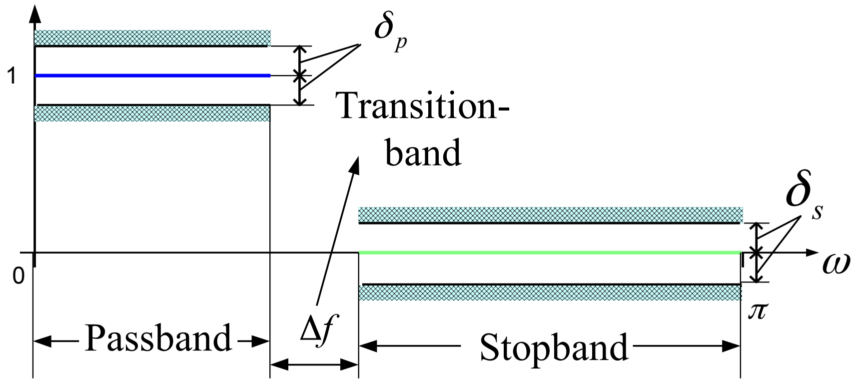
CSE4210 Architecture & Hardware for DSP

Basics of Digital Filter

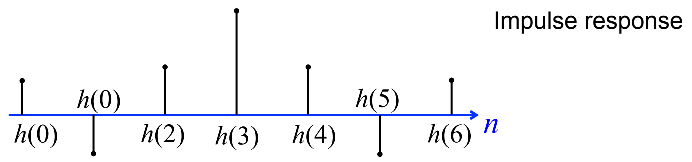
Filter Specifications

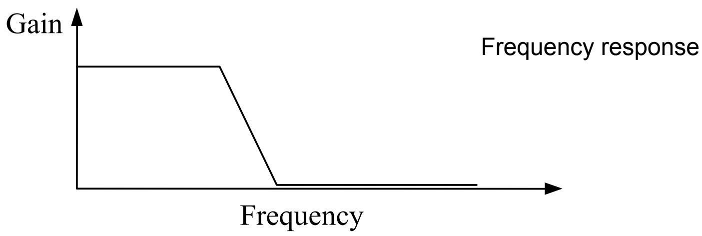


Finite Impulse Response (FIR) Filters

- Example of an FIR filter

$$y(n) = h(0)x(n) + h(1)x(n-1) + \cdots + h(6)x(n-6)$$





Representation of an FIR filter

- By convolution sum

$$y(n) = h(0)x(n) + h(1)x(n-1) + \cdots\cdots + h(N-1)x(n-N+1)$$

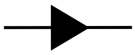
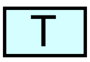
$$= \sum_{m=0}^{N-1} h(m)\; x(n-m)$$

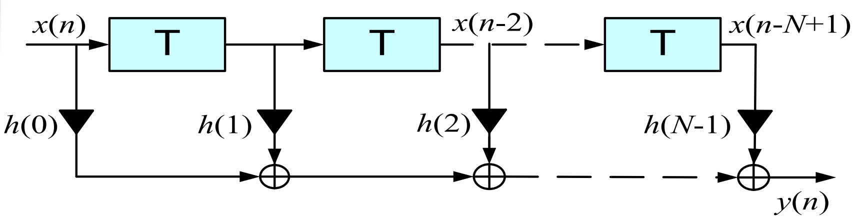
- By *z*-transform transfer function

$$H(z) = \sum_{m=0}^{N-1} h(m)\; z^{-m}$$

$$H\left(e^{j\omega}\right) = \sum_{m=0}^{N-1} h(m) e^{-j\omega m}\ ,\ \omega = 2\pi f T$$

Implementation of FIR Filters

- Three main components:
  - Adder – $\oplus$
  - Multiplier – 
  - Delay – 



$h(n)$, n=0,…,N-1, are coefficients.

Demonstrations of FIR Filters

- Let us consider a low pass FIR filter,

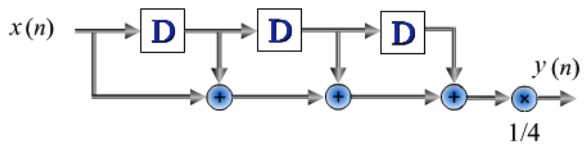
$$y(n) = \frac{1}{4}\left[x(n) + x(n-1) + x(n-2) + x(n-3)\right] \qquad (1)$$

Its *z*-transform transfer function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4}\left[1 + z^{-1} + z^{-2} + z^{-3}\right] \qquad (2)$$

How Does an FIR Filter Works?

$$y(n) = \frac{1}{4}\left[x(n) + x(n-1) + x(n-2) + x(n-3)\right] \qquad (1)$$



Click to begin demo

Frequency Response

- Consider a complex exponential input sequence

$$x(n) = e^{j\omega n} \qquad -\infty < n < \infty$$

- If the impulse response of the system is *h*(*n*), the output is :

$H(e^{j\omega})$ is called frequency response

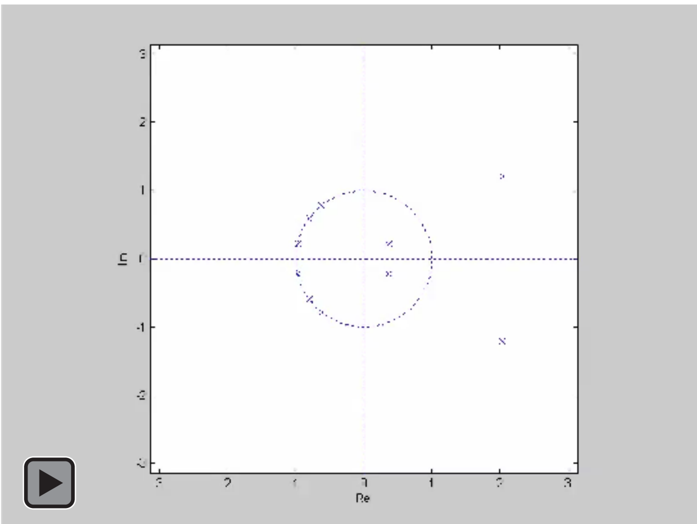
$$y(n) = \sum_{m=-\infty}^{\infty} h(m) e^{j\omega(n-m)} = e^{j\omega n} \sum_{m=-\infty}^{\infty} h(m) e^{-j\omega m}$$

$$= H(e^{j\omega}) x(n)$$

$$H(e^{j\omega}) = \sum_{m=-\infty}^{\infty} h(m) e^{-j\omega m} = H(z)\big|_{z=e^{j\omega}}$$

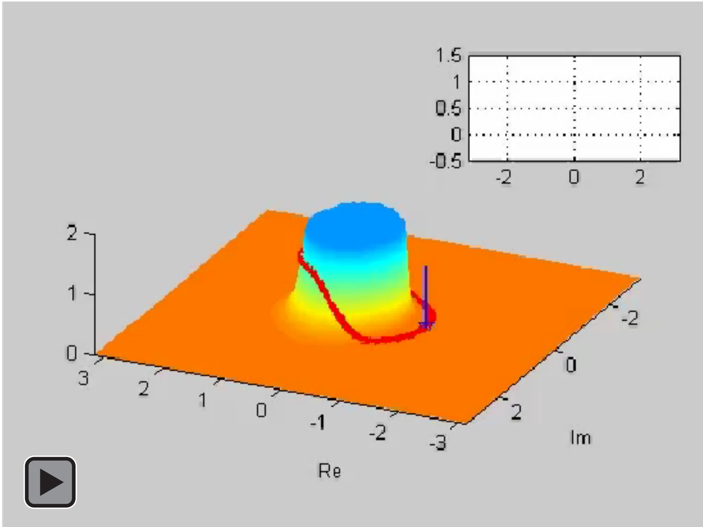
Demonstration 1

z-plane Representation of an FIR Filter



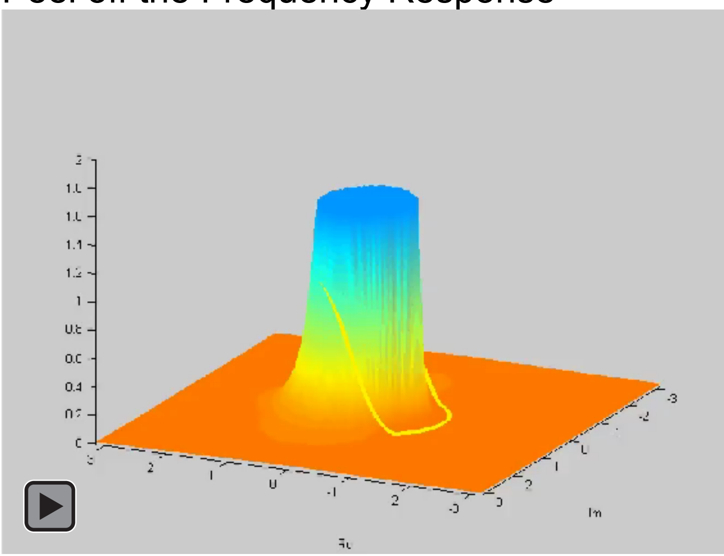
Demonstration 2

Frequency Response in a *z*-plane



Demonstration 3

Peel off the Frequency Response



Compute Frequency Response

- Magnitude and phase response

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\theta(\omega)}$$

Magnitude response | Phase response

- Compute frequency response using Matlab

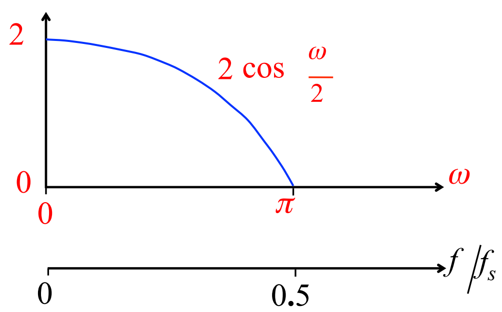
[H,w]=freqz(b,a,N);
-- returns the N-point frequency vector w in radians and the N-point complex frequency response vector H of the B(z)/A(z).

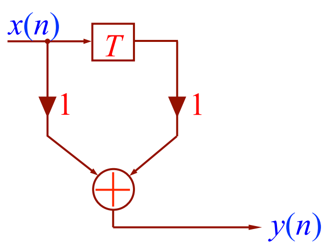
$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b(k)z^{-k}}{\sum_{k=0}^{p} a(k)z^{-k}}$$

Example 1: $H(z) = 1 + z^{-1}$

$$H(e^{j\omega}) = 1 + e^{-j\omega} = 2\, e^{-j\frac{\omega}{2}} \frac{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}{2}$$

$$= 2\, e^{-j\frac{\omega}{2}} \quad \cos\frac{\omega}{2}$$

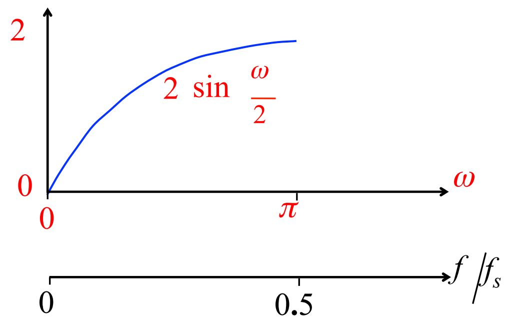


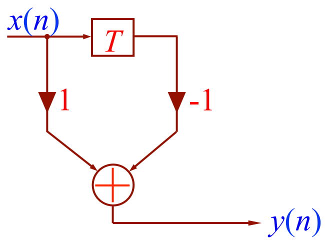


Example 2 : $H(z) = 1 - z^{-1}$

$$H(e^{j\omega}) = 1 - e^{-j\omega} = e^{j\frac{\pi}{2}} \ e^{-j\frac{\omega}{2}} \ 2 \ \frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{2j}$$

$$= 2 \ e^{j(\frac{\pi}{2} - \frac{\omega}{2})} \ \sin\frac{\omega}{2}$$





Properties of FIR Filter

1. $\left|H(e^{j\omega})\right| = \left|H(e^{-j\omega})\right|$

2. $\angle H(e^{j\omega}) = -\angle H(e^{-j\omega})$

Proof: $H(e^{-j\omega}) = \sum_{m=0}^{N-1} h(m)\, e^{j\omega m}$

$$= \left\{ \sum_{m=0}^{N-1} h(m)\, e^{-j\omega m} \right\}^{*} , \quad h(m) \ \text{real}$$

$$= H^{*}(e^{j\omega})$$

Properties of FIR Filter

3. $H(e^{j\omega}) = H(e^{j(\omega+2\pi m)})$

$$\text{Proof}: \quad H(e^{j(\omega+2m\pi)}) = \sum_{n=0}^{N-1} h(n)\, e^{-j(\omega+2m\pi)n}$$

$$= \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = H(e^{j\omega})$$

For $h(n)$ real, knowledge of $H(e^{j\omega})$ between $\omega = 0$ and $\omega = \pi \Rightarrow$ *knowledge of* $H(e^{j\omega})$ ω for any ω.

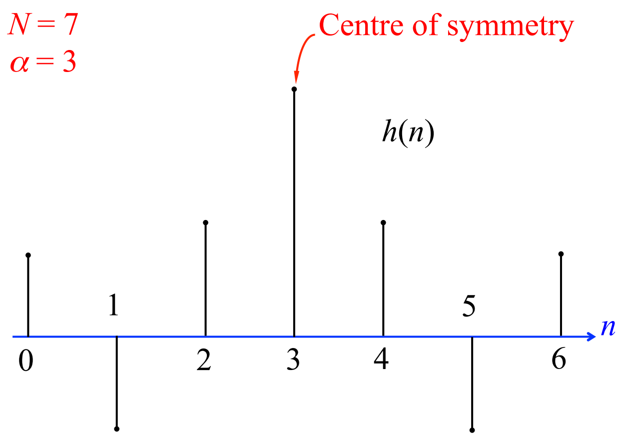
Linear Phase FIR Filter

- An FIR filter may be designed to have linear phase characteristics.
- The phase response of a linear phase FIR filter is either $-\alpha\omega$ or $\beta-\alpha\omega$ where $\alpha = (N-1)/2$, ω is the frequency, $\beta = \pm 0.5\pi$, and N is the filter length.
- Its frequency response is given by $e^{-j\frac{N-1}{2}\omega} R(\omega)$ or $e^{j\frac{\pi}{2}-j\frac{N-1}{2}\omega} R(\omega)$, where $R(\omega)$ is a real function.

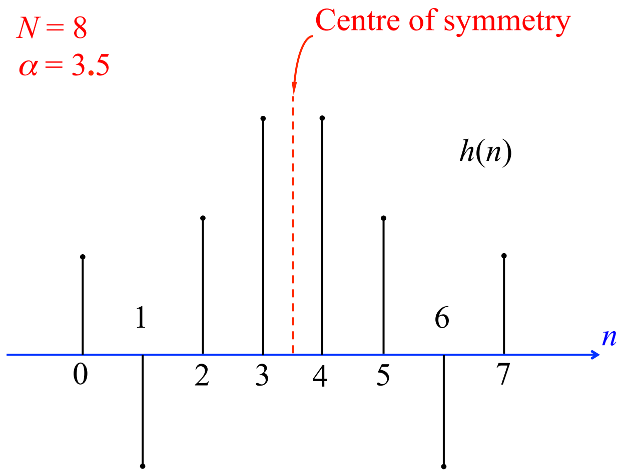
Linear Phase FIR Filter

- Its impulse response is either symmetrical or anti-symmetrical.
- If its impulse response is symmetrical, its phase response is $-\alpha\omega$.
- If its impulse response is anti-symmetrical, its phase response is $\beta - \alpha\omega$.

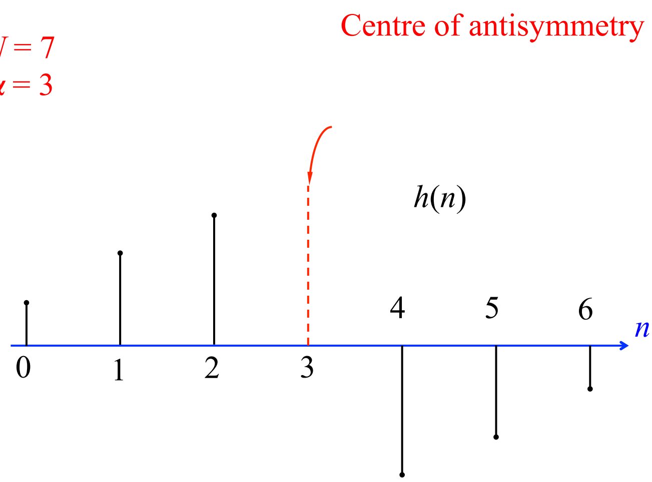
Symmetrical Impulse Response, *N* odd



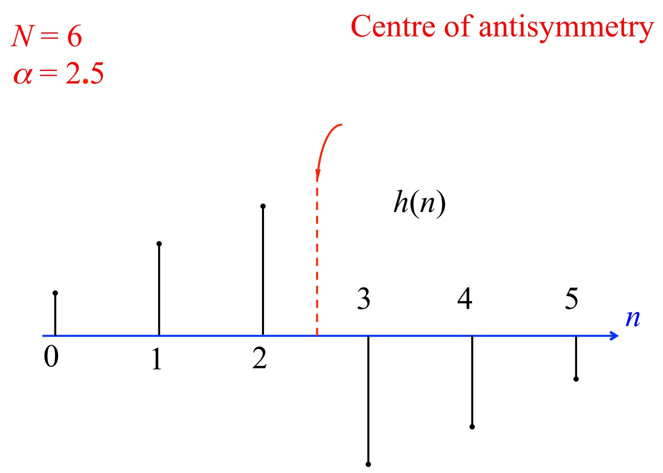
Symmetrical Impulse Response, *N* even



Anti-symmetrical Impulse Response, *N* odd



Anti-symmetrical Impulse Response, *N* even



Frequency Response of Linear Phase FIR Filter

- 4 types, depending on whether *N* is odd or even and whether the impulse response is symmetrical or anti-symmetrical.

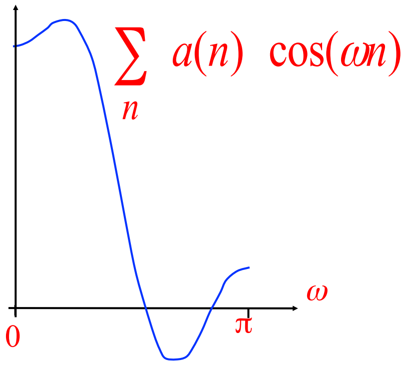
Type 1: Symmetrical Impulse Response, *N* odd.

$$H(e^{j\omega}) = e^{-j\omega \frac{N-1}{2}} \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos(\omega\, n)$$

$$a(0) = h\left(\tfrac{N-1}{2}\right)$$

$$a(n) = 2\,h\left(\tfrac{N-1}{2} - n\right),$$

$$n = 1,\ 2,\ \cdots,\ \tfrac{N-1}{2}$$



Proof:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\tfrac{N-1}{2}\right)e^{-j\omega \frac{N-1}{2}} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)e^{-j\omega n}$$

$$= e^{-j\omega \frac{N-1}{2}} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n)\left\{e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right\} + h\left(\tfrac{N-1}{2}\right)\right]$$

$$= e^{-j\omega \frac{N-1}{2}} \left\{ \sum_{n=0}^{\frac{N-3}{2}} 2h(n)\cos\left[\omega\left(\tfrac{N-1}{2} - n\right)\right] + h\left(\tfrac{N-1}{2}\right)\right\}$$

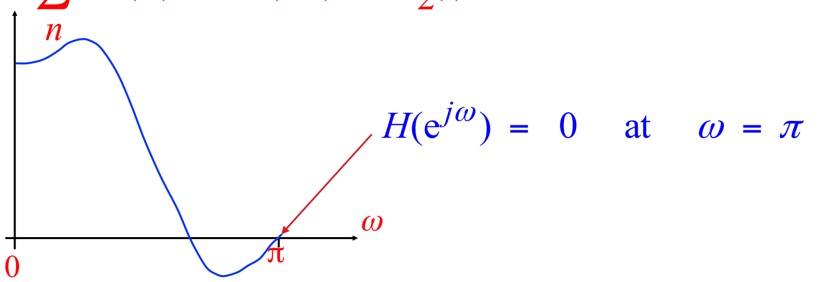
$$= e^{-j\omega \frac{N-1}{2}} \left\{ \sum_{m=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2} - m\right) \cos\omega\, m + h\left(\tfrac{N-1}{2}\right)\right\}$$

Type 2 :Symmetrical Impulse Response, *N* even

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \sum_{n=1}^{\frac{N}{2}} b(n)\cos\left(\omega(n-\frac{1}{2})\right)$$

$$b(n) = 2h(\tfrac{N}{2}-n),\ n = 1,2,\ldots,\tfrac{N}{2}$$

$$\sum_n b(n)\ \cos(\omega(n-\tfrac{1}{2}))$$

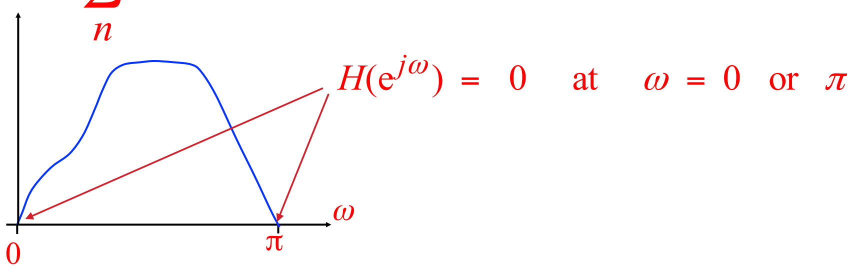


Type 3:Anti-symmetrical Impulse Response, *N* odd

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \sum_{n=0}^{\frac{N-1}{2}} c(n)\sin(\omega n),\ h\left(\frac{N-1}{2}\right) = 0$$

$$c(n) = 2h\left(\frac{N-1}{2}-n\right),\ n = 1,2,\ldots,\frac{N-1}{2}$$

$$\sum_n c(n)\sin(\omega n)$$



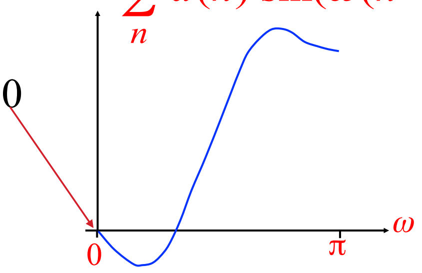
Type 4:Anti-symmetrical Impulse Response, *N* even

$$H(e^{j\omega}) = e^{-j\frac{N-1}{2}\omega} e^{j\frac{\pi}{2}} \sum_{n=1}^{\frac{N}{2}} d(n)\sin\left(\omega(n-\tfrac{1}{2})\right)$$

$$d(n) = 2h(\tfrac{N}{2}-n),\ n = 1, 2, \cdots, \tfrac{N}{2}$$

$$\sum_n d(n)\sin(\omega(n-\tfrac{1}{2}))$$

$$H(e^{j\omega}) = 0\ at\ \omega = 0$$



FIR filter length estimation

$$L = \frac{-20\log(\sqrt{\delta_p\delta_s}) - 13}{14.6\Delta f} + 1$$

$\delta_p < 1$, Passband ripple,

$\delta_s < 1$, Stopband ripple/attenuation

Δf = Normalized transition-width

= |stopband edge - passband edge|

Filter length and complexity

- FIR filter transfer function:

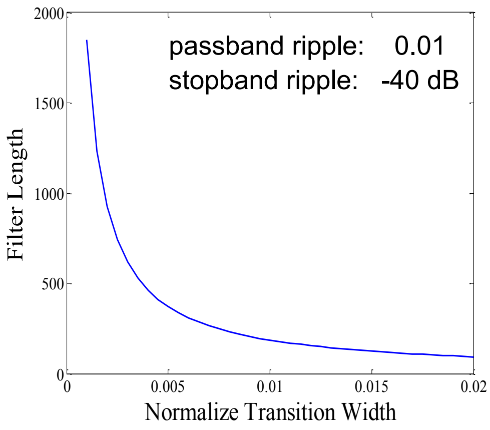
$$H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}$$

- Filter length=the order of transfer function +1.
- Complexity=No. of taps (coefficients) for a filter.
- For a symmetric filter, the filter complexity is about the half of the filter length.

Complexity of a FIR Filter

$$L = \frac{-20\log(\sqrt{\delta_p \delta_s}) - 13}{14.6\Delta f} + 1$$

where δ_p and δ_s are passband and stopband ripple; Δf is the transition width.



FIR Filters

- Advantages :
  - Exact linear-phase characteristic.
  - Intrinsically stable implementation.
- Disadvantages :
  - Require a high-order transfer function compared with infinite-duration impulse response filters.

FIR Filter Design

- Windowing
- Frequency sampling
- Weighted Chebyshev approximation
- Demos are available in e-Learning Hub http://elearninghub.eng.nus.edu.sg under “simulation” → “Virtual simulation 5”

Parks-McClellan Optimal Equiriple FIR Filter Design Using Matlab

- Matlab functions for design FIR filter: “firpmord” and “firpm”.
- How to use the functions:
  - [N,Fi,Ai,W]=firpmord(F,A,Dev,Fs);
  - B=firpm(N,Fi,Ai,W) returns the coefficients of the resulting FIR filter which has the best approximation to the desired frequency response described by F, A, and Dev, where
    - F is a vector of filter bandedges in Hz.
    - A is a real vector indicate the desired amplitude on the bands defined by F.
    - Dev is a vector of maximum deviations or ripples allowable for each band. Dev must have the same length as A.
    - Fs is the sampling frequency.

Example: a lowpass filter with fpass=1500Hz, fstop=2000Hz,fsample=8000Hz, rp=rs=0.01

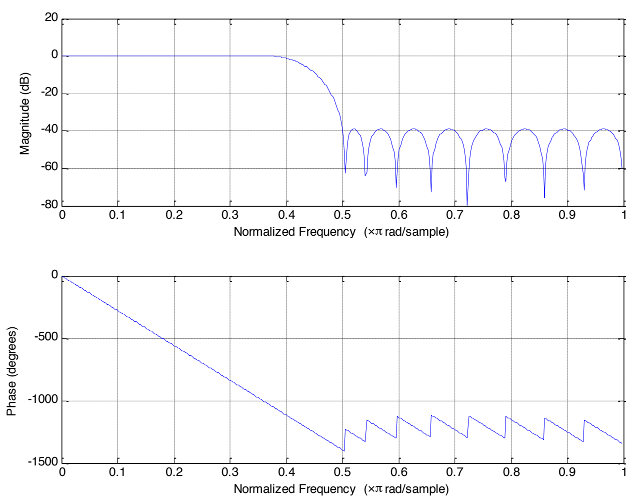
- F=[1500, 2000];A=[1,0]; Dev=[0.01,0.01]
- Using Matlab command “firpmord” to estimate the filter length.

  [N,Fi,Ai,W]=firpmord(F,A,Dev,8000);
- Find the coefficients: B=firpm(N,Fi,Ai,W);
- Plot frequency response : freqz(B,1);
- Need help: type “help firpm” in Matlab.

Coefficients

h(0)=   0.0029=h(31)
h(1)=   0.0094=h(30)
h(2)= -0.0037=h(29)
h(3)= -0.0109=h(28)
h(4)= -0.0014=h(27)
h(5)=   0.0167=h(26)
h(6)=   0.0100=h(25)
h(7)= -0.0204=h(24)
h(8)= -0.0249=h(23)
h(9)=   0.0190=h(22)
h(10)= 0.0479=h(21)
h(11)= -0.0064=h(20)
h(12)= -0.0855=h(19)
h(13)= -0.0358=h(18)
h(14)= 0.1853=h(17)
h(15)= 0.4033=h(16)

Frequency Response



References

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3. DSP First: A Multimedia Approach, by J.H. McClellan, R.W. Schafer, and M.A. Yoder, Prentice Hall.
4. Digital Signal Processing, By A.V. Oppenheim, R.W. Schafer, Prentice-Hall.