ENG2200
Electric Circuits

Chapter 8

RLC Circuit

Natural and Step Response

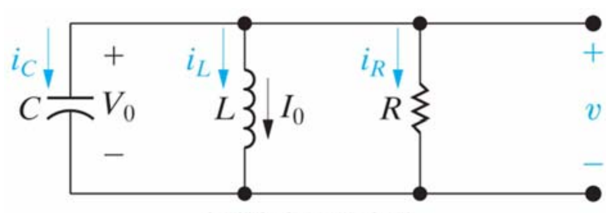
ENG2200 Topics to be covered

- Be able to determine the natural response of RLC circuits
- Be able to determine the step response of RLC circuits.

RLC Circuits

- The first step is to write either KVL or KCL for the circuit.
- Take the derivative to remove any integration
- Solve the resulting differential equation

Parallel RLC circuit



KCL
Sum of currents =0

$$i_R + i_L + i_c = 0$$

$$\frac{v}{R} + \frac{1}{L}\int_0^t v\,d\tau + I_0 + C\frac{dv}{dt} = 0$$

Differentiating with respect to t, we get

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{v}{L} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

Solve this equation

Parallel RLC Circuits

- How to solve this differential equation?
- We can not separate the variables like we did with the RC or RL circuits.
- Without going into a lot of Math, we claim the solution will be in the form $v=Ae^{st}$
  - Exponential is the only function where high order of the derivatives have the same form (exponential)
  - First order (RL or RC) have the same form

Parallel RLC Circuits

- Assuming $v = Ae^{st}$ and substituting in the equation

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

$$v = Ae^{st}$$

$$As^2e^{st} + \frac{As}{RC}e^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st}\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right) = 0$$

Either A=0 (trivial solution) or
The quadratic part is 0

Characteristic equation

Parallel RLC Circuits

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Both of these 2 values satisfy the equation, their sum does.

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

With a special case when $s_1 = s_2$

The roots are determined by R,L, and C.
The initial conditions determine A_1, A_2

Parallel RLC Circuits

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad , \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

Neper frequency (α) — Resonant radian frequency (ω_0) — Damping ratio (ξ)

Parallel RLC Circuits

- The solution of the differential equation depends on the values of s1 and s2
- For simplicity assume A_1 and A_2 to be 1

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- Consider three cases

Overdamping $\alpha > \omega_o$, $\zeta > 1$ -- 2 real roots

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

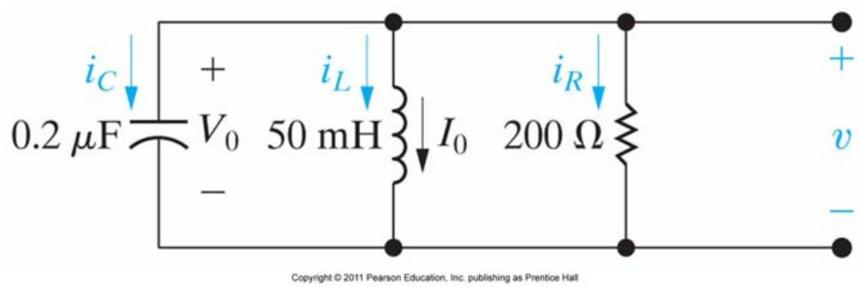
$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad , \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

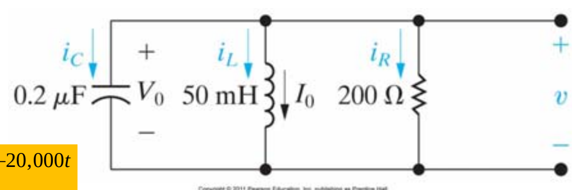
$$v(t) = A_1 e^{-s_1 t} + A_2 e^{-s_2 t}$$

Parallel RLC Circuits



$$v_C(0^-) = v_C(0) = v_C(0^+)$$
$$i_L(0^-) = i_L(0) = i_L(0^+)$$

Example



$$v(t) = A_1 e^{-5,000t} + A_2 e^{-20,000t}$$

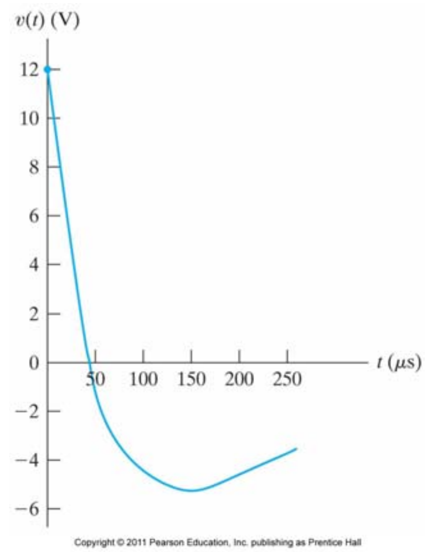
$$v(0^+) = 12 = A_1 + A_2$$

$v(0^+)=12\ V$
$i_L(0^+)=30\ mA$

$$i_C(0^+) = -i_R(0^+) - i_L(0^+)$$
$$i_C(0^+) = C\frac{dv(0^+)}{dt}$$
$$C\frac{dv(0^+)}{dt} = -\frac{V_0}{R} - i_L(0^+)$$
$$\frac{dv(0^+)}{dt} = -\frac{V_0}{CR} - \frac{i_L(0^+)}{C}$$

$$\frac{dv(0^+)}{dt} = -450,000V/s = -5000A_1 - 20,000A_2$$

2 equation in 2 unknowns, solve to get A1 and A2



Underdamping $\alpha < \omega_o$, $\zeta < 1$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

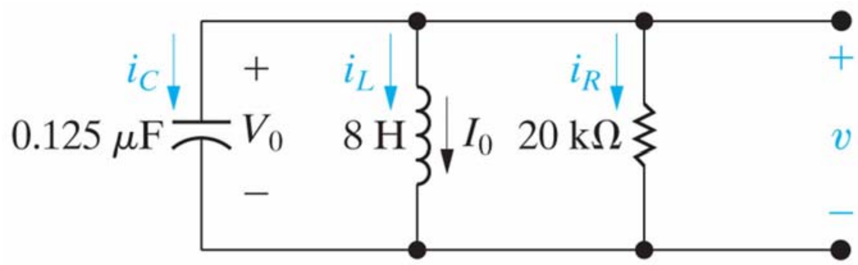
$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad , \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

$$s = -\alpha \pm j\omega_d \qquad \omega_d = \sqrt{\omega_0^2 - \alpha}$$

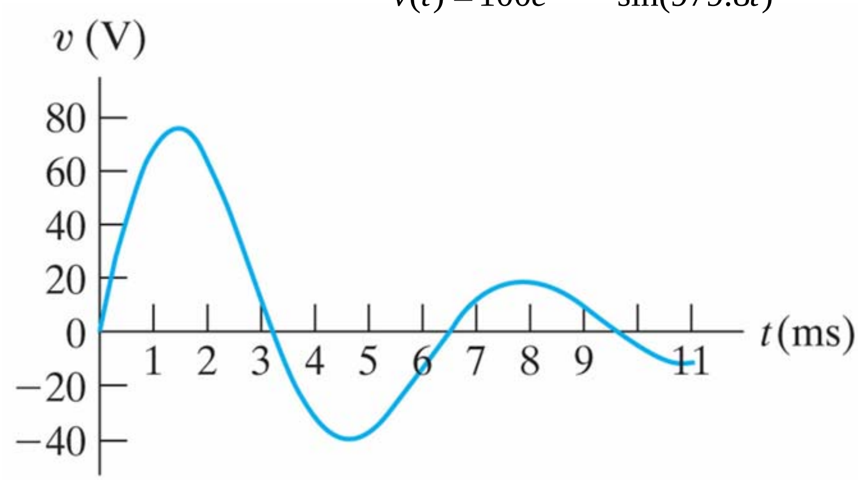
$$v(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t)$$

Example



V0=0
I0=-12.25mA

$$v(t) = 100e^{-200t}\sin(979.8t)$$



Critically Damped $\alpha=\omega_o$, $\zeta= 1$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

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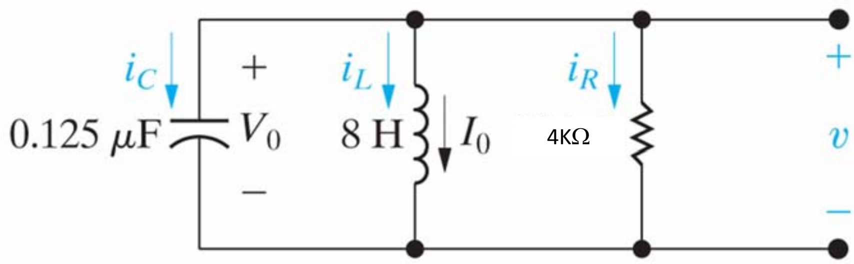
$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

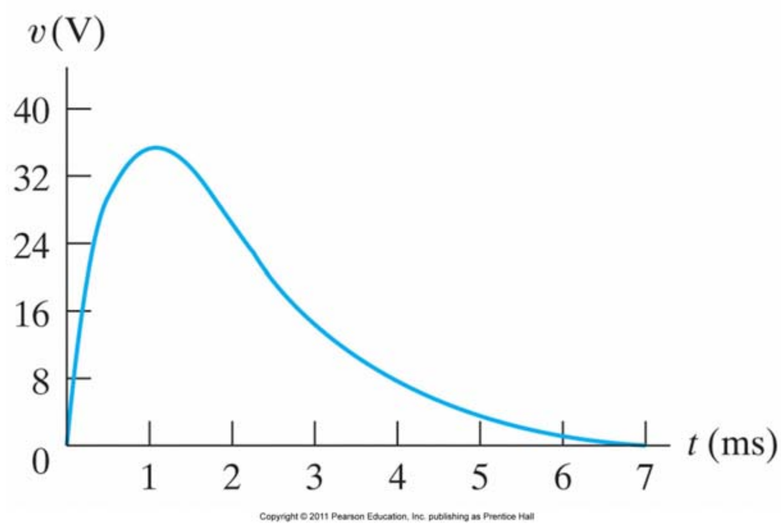
$$\omega_0 = \frac{1}{\sqrt{LC}} \quad , \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

$$s_1 = s_1 = -\alpha$$

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

Example



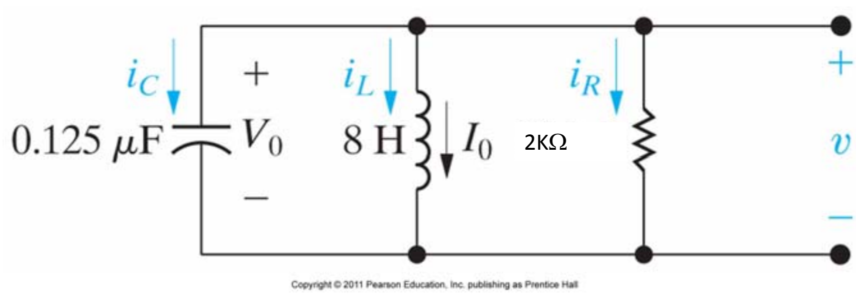


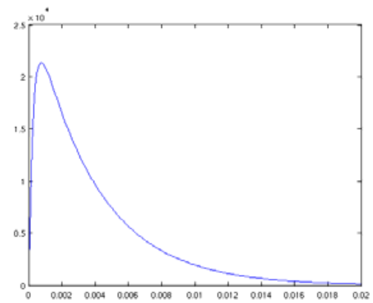
$\xi = 2$

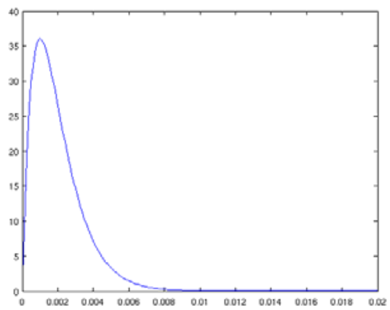
$\alpha = 2000, \quad \omega_0 = 1000$

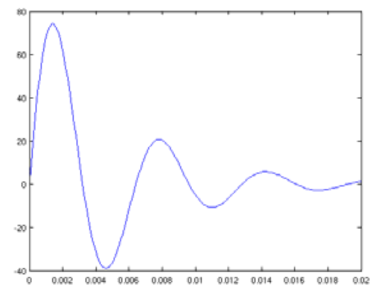
$s_1 = -267.95$

$s_2 = 3732$



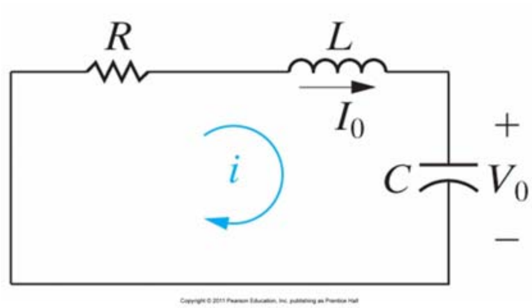




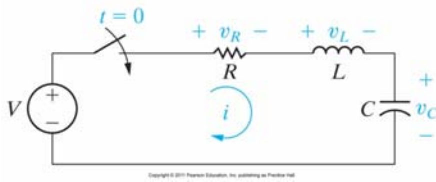


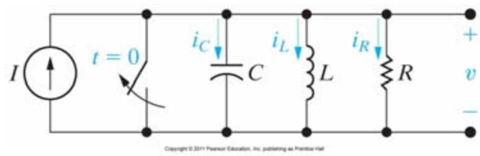
Series RLC

- Solved in the lab manual
- Only difference is α=R/2L



Step Response of RLC Circuit





$$v_L + v_C + v_R = V$$

$$L\frac{di}{dt} + v_c + iR = V$$

$$i = C\frac{dv_C}{dt}$$

$$LC\frac{d^2 v_C}{dt^2} + v_C + RC\frac{dv_C}{dt} = V$$

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{I}{LC}$$

$$i_L + i_C + i_R = I$$

$$i_L + C\frac{dv}{dt} + \frac{v}{R} = I$$

$$v = L\frac{di_L}{dt}$$

$$LC\frac{d^2 i_L}{dt} + \frac{L}{R}\frac{di_L}{dt} + i_L = I$$

$$\frac{d^2 i_L}{dt} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

Step Response of RLC Circuits

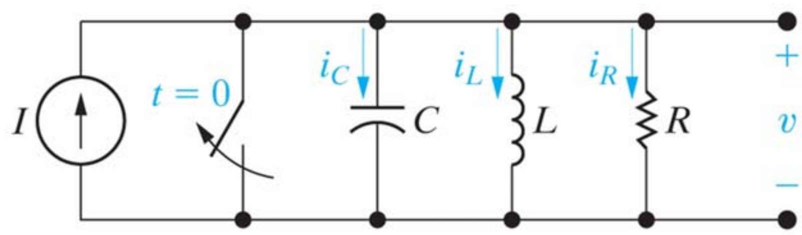
- A topic for a course in Math.
- Generally speaking, the solution of a second-order DE with a constant driving force equals the forced response plus the a response function identical to the natural response.

$$i = I_f + \begin{Bmatrix} \text{function of the same form} \\ \text{as natural response} \end{Bmatrix}$$

$$v = V_f + \begin{Bmatrix} \text{function of the same form} \\ \text{as natural response} \end{Bmatrix}$$

Example

- What is the final I_f



Example

- What is the final V_f

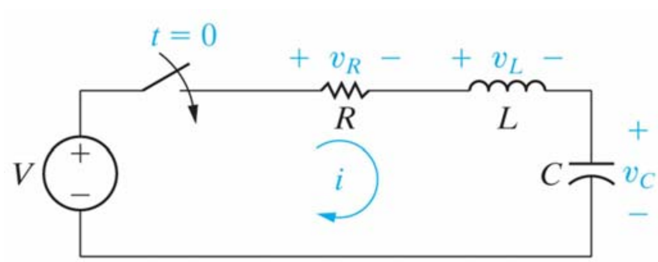
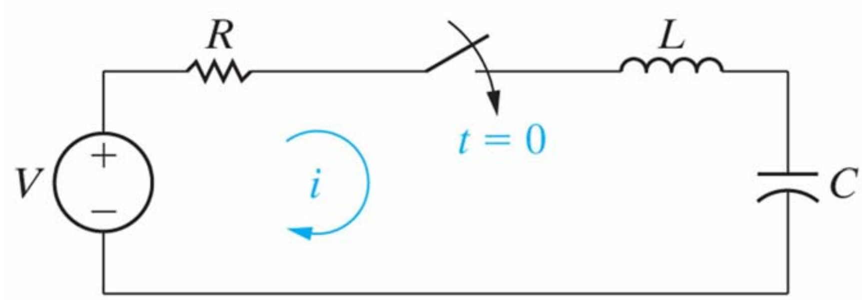


Figure 8.4 A circuit used to illustrate the step response of a series *RLC circuit.*



$v_c = V_f + A_1 e^{s_1 t} + A_2 e^{s_2 t}$	Overdamped
$v_c = V_f + B_1 e^{\alpha t} \cos \omega_d t + B_2 e^{\alpha t} \sin \omega_d t$	Underdamped
$v_c = V_f + D_1 e^{\alpha t} + D_2 e^{\alpha t}$	Critically damped

TABLE 8.1 Natural Response Parameters of the Parallel *RLC* Circuit

Parameter	Terminology	Value In Natural Response
s_1, s_2	Characteristic roots	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
α	Neper frequency	$\alpha = \frac{1}{2RC}$
ω_0	Resonant radian frequency	$\omega_0 = \frac{1}{\sqrt{LC}}$

Figure 8.12 The circuit for Example 8.6.

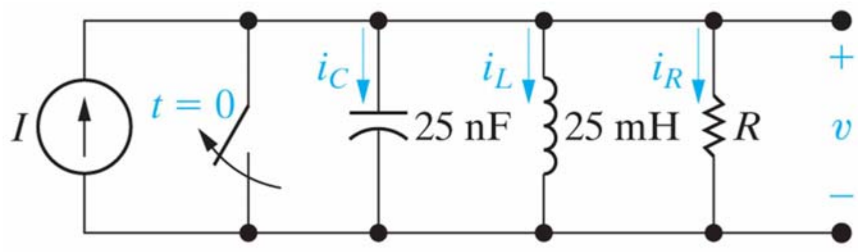


Figure 8.13 The current plots for Example 8.9.