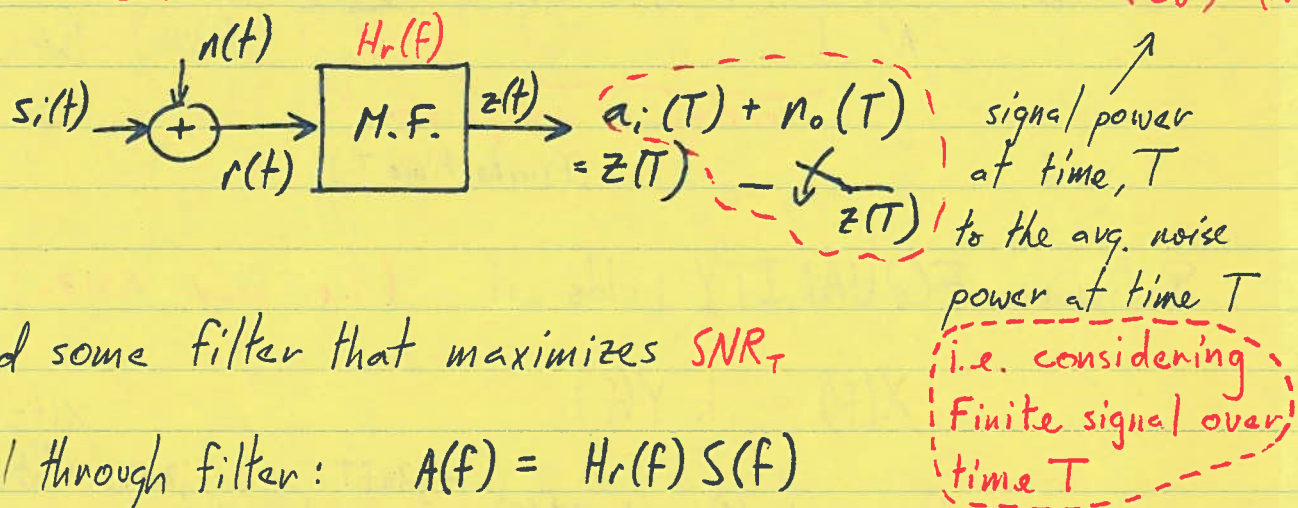


L10 Matched Filtering & Optimum Detection

10.1 Matched Filter

• For ML knowing $P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$ we are looking to maximize $\frac{a_1 - a_2}{2\sigma_0}$ $\max(a_1 - a_2)$ is obvious

but $\max\left(\frac{a_i}{\sigma_0}\right)$ is also important, which is prop. to $\max\left(\frac{a_i^2}{\sigma_0^2}\right) = \left(\frac{S}{N/T}\right)$



• build some filter that maximizes SNR_T

• signal through filter: $A(f) = H_r(f) S(f)$

$$a(t) = \text{F.T.}\{A(f)\} = \int_{-\infty}^{\infty} H_r(f) S(f) e^{j2\pi ft} df$$

• noise through filter (PSD)

$$N_o(f) = \frac{N_0}{2} |H_r(f)|^2$$

$$\sigma_0^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} |H_r(f)|^2 df$$

$$SNR_T = \frac{\left| \int_{-\infty}^{\infty} H_r(f) S(f) e^{j2\pi ft} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H_r(f)|^2 df}$$

• to keep going invoke Schwarz Inequality

$$\left| \int_{-\infty}^{\infty} X(f) Y^*(f) df \right|^2 \leq \int_{-\infty}^{\infty} |X(f)|^2 df \int_{-\infty}^{\infty} |Y(f)|^2 df$$

$$\therefore \text{SNR}_T \leq \frac{\int_{-\infty}^{\infty} |H_r(f)|^2 df \int_{-\infty}^{\infty} |S(f) e^{-j2\pi f T}|^2 df}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H_r(f)|^2 df}$$

$$\text{SNR}_T \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |S(f)|^2 df = \frac{2 E_b}{N_0} \frac{[J]}{[W/Hz]} = \frac{[J]}{[W \cdot s]} = \frac{[J]}{[J]}$$

energy of energy signal
(finite time T)

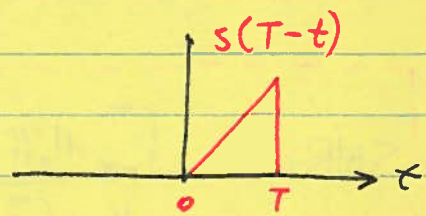
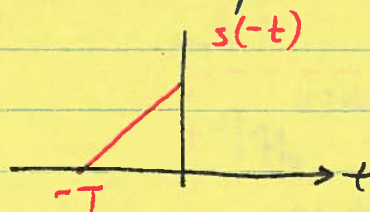
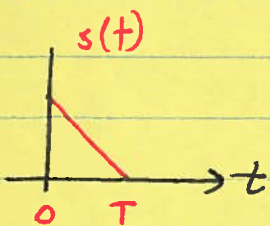
Schwarz EQUALITY holds if (i.e. max. SNR_T)

$$X(f) = k Y(f)$$

$$\therefore \text{setting } H_r(f) = k S^*(f) e^{-j2\pi f T} = k S(-f) e^{j2\pi f T} \quad \begin{matrix} x(t-t_0) \rightarrow X(f) e^{-j2\pi f t_0} \\ X^*(f) = X(-f) \\ x(-t) \leftrightarrow X(-f) \end{matrix}$$

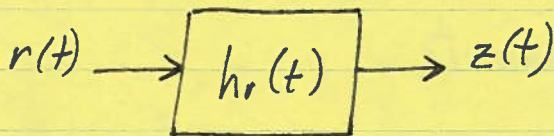
$$h_r(t) = \text{F.T.}^{-1} \{ H_r(f) \} = k s(T-t) = h_r(t)$$

∴ receive (matched) filter impulse response is mirror image of s(t) delayed by T



10.2 Correlator Equivalent

- Another way to view the M.F.



$$z(t) = \int_{-\infty}^{\infty} r(t-\tau) h_r(\tau) d\tau$$

- for causal system if i/p starts at $t=0$

$$z(t) = \int_0^{\infty} r(t-\tau) h_r(\tau) d\tau \quad \left(z(t) = \int_0^t r(t-\tau) h(\tau) d\tau \right)$$

- w/i change of variables

$$\left. \begin{array}{l} s = t - \tau \quad \tau = t - s \\ ds = -d\tau \quad \tau|_0 = s|_t \\ \tau|_{\infty} = s|_{-\infty} \end{array} \right\} z(t) = \int_{-\infty}^{\infty} r(s) h(t-s) (-ds) \\ = \int_{-\infty}^t r(s) h(t-s) ds$$

$$\downarrow z(t) = \int_{-\infty}^t r(\tau) h_r(t-\tau) d\tau \quad \left(z(t) = \int_0^t r(\tau) h(t-\tau) d\tau \right) \downarrow$$

- substitute MF impulse response

$$z(t) = \int_0^t r(\tau) s(T - (t - \tau)) d\tau$$

$$z(t) = \int_0^t r(\tau) s(T - t + \tau) d\tau$$

at $t = T \dots \rightarrow$

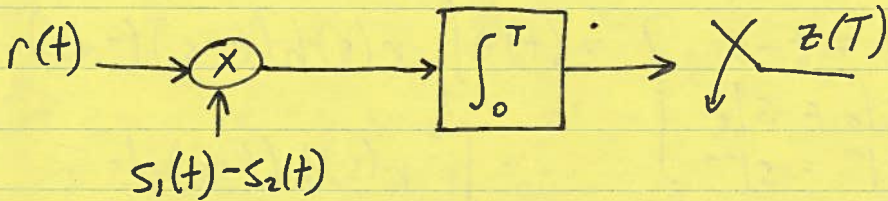
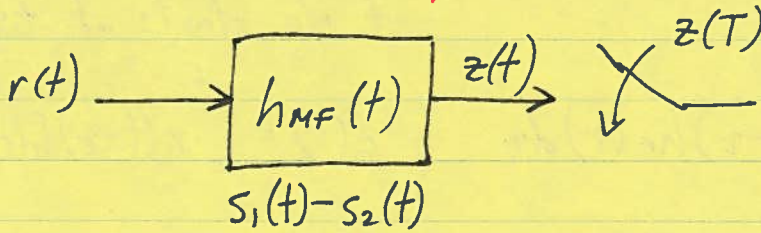
$$z(T) = \int_0^T r(\tau) s(\tau) d\tau \quad \text{recall } R_x(\tau) = \int_{-\infty}^{\infty} x(t) x(t+\tau) dt$$

a cross-correlation

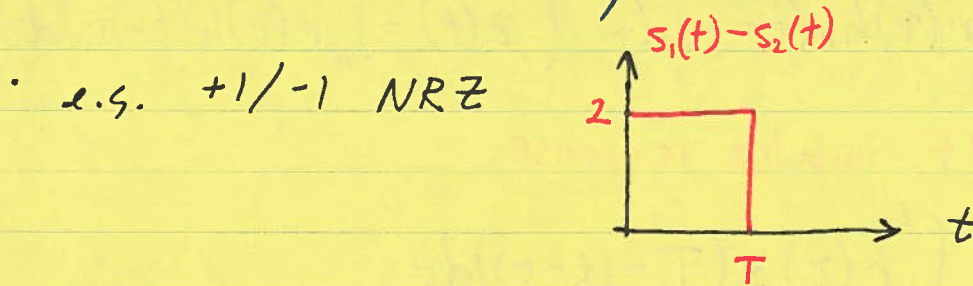
$$R_x(0) = \int_{-\infty}^{\infty} x(t) x(t) dt$$

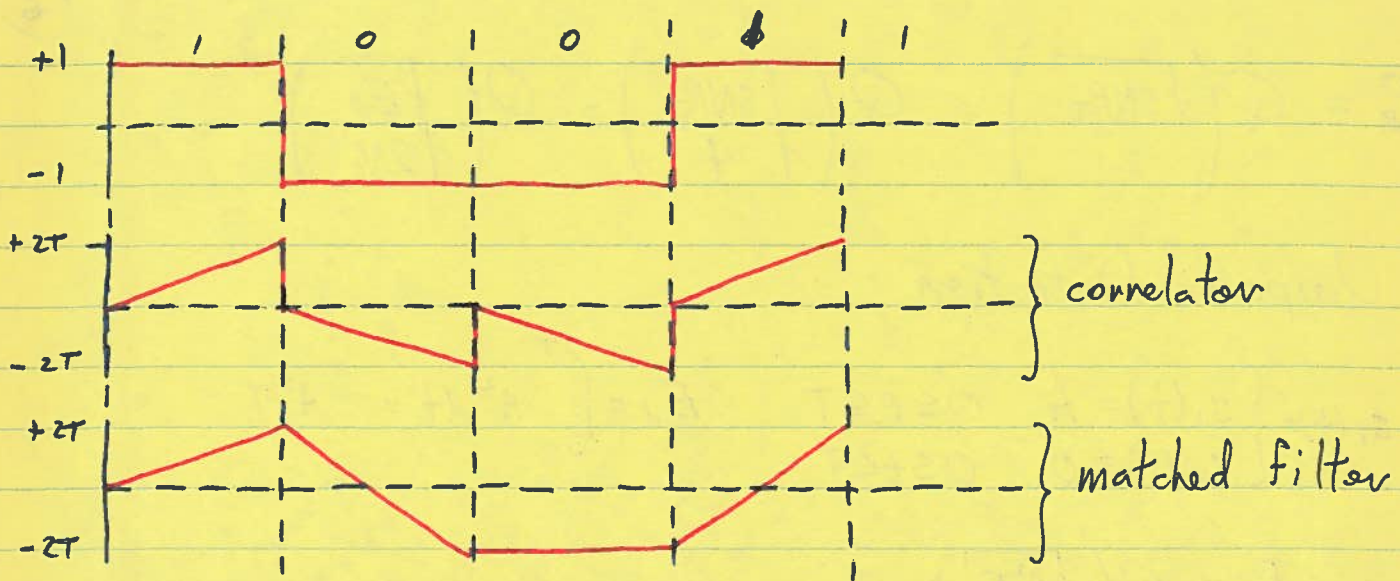
10.3 Receive Filter Arrangements

- looking to max. $\frac{a_1 - a_2}{b_0}$
- so match to the *signal difference*



- *MF and correlator* only match results at $t=T!!!$





- correlator is a little easier to make (multiplier-accumulator units) and can be a little more forgiving of timing errors in bandpass systems
- matched filter is easier to realize as part of a more complex system that merges a number of filtering functions (e.g. whitening & equalization)

10.4 Optimum Detection

• ML-D combined with matched filter/correlator results in optimum detection

to minimize $P_B = Q\left(\frac{a_1 - a_2}{2b_0}\right)$ we maximize $SNR_T = \frac{(a_1 - a_2)^2}{b_0^2}$

inst. pwr. @ T
↓
ratio of avg. power

• through MF/corr. SNR_T we showed this power ratio is equivalent to the energy ratio

energy of diff signal $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt$

$$SNR_T = \frac{(a_1 - a_2)^2}{b_0^2} = \frac{E_d}{N_0/2}$$

$$P_B = Q\left(\sqrt{\frac{SNR_T}{2}}\right) = Q\left(\sqrt{\frac{SNR_T}{4}}\right) = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

10.5 Unipolar Detection

$$s_i(t) = \begin{cases} s_1(t) = A & 0 \leq t \leq T \\ s_2(t) = 0 & 0 \leq t \leq T \end{cases} \quad E_d = \int_0^T A^2 dt = A^2 T$$

$$\therefore P_B = Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right)$$

- in terms of avg. bit energy?

$$E_b = \frac{1}{2} E_1 + \frac{1}{2} E_2 = \frac{1}{2} \int_0^T A^2 dt + \frac{1}{2} \int_0^T 0 dt = \frac{A^2 T}{2}$$

$$\therefore P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

10.6 Bipolar Detection

$$s_i(t) = \begin{cases} s_1(t) = +A & E_d = \int_0^T 4A^2 dt = 4A^2 T \\ s_2(t) = -A \end{cases}$$

$$\therefore P_B = Q\left(\sqrt{\frac{2A^2 T}{N_0}}\right)$$

$$E_b = \frac{1}{2} A^2 T + \frac{1}{2} A^2 T = A^2 T$$

$$\therefore P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \leftarrow \text{which is better}$$

10.7 Waterfall Curves

