CSE 2021 Computer Organization

# Chapter 3

Arithmetic for Computers

## Arithmetic for Computers

- Operations on integers
  - Addition and subtraction
  - Multiplication and division
  - Dealing with overflow
- Floating-point real numbers
  - Representation and operations

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## Arithmetic Operations on Integers

## Integer Addition & Subtraction

- Addition example: 7 + 6

| | | | | | | |
|---|---|---|---|---|---|---|
| | (0) | (0) | (1) | (1) | (0) | (Carries) |
| . . . | 0 | 0 | 0 | 1 | 1 | 1 |
| . . . | 0 | 0 | 0 | 1 | 1 | 0 |
| . . . (0) | 0 (0) | 0 (0) | 1 (1) | 1 (1) | 0 (0) | 1 |

- Subtraction example: 7-6=7+(-6)
  - Add negation of second operand

```
+7:   0000 0000 … 0000 0111
–6:   1111 1111 … 1111 1010   2's complement
+1:   0000 0000 … 0000 0001
```

## Addition of Signed Numbers

More examples below are shown for 4-bit 2's complement arithmetic.

| | | | |
|---|---|---|---|
| 1. | (+5) | 0101 | |
| | +(+2) | +0010 | |
| | (+7) | 0111 | |
| 2. | (-5) | 1011 | |
| | +(+2) | +0010 | |
| | (-3) | 1101 | |
| 3. | (+5) | 0101 | |
| | +(-2) | +1110 | |
| | (+3) | 1 0011 | ignore the carry |
| 4. | (-5) | 1011 | |
| | +(-2) | +1110 | |
| | (-7) | 1 1001 | ignore the carry |

## Overflow

- Example: 7 + 6 (each number in signed 4-bit )

```
+ 7:    0111
+ 6:    0110
+13:    1101 → -3
```

**Overflow**

- Overflow if result out of range

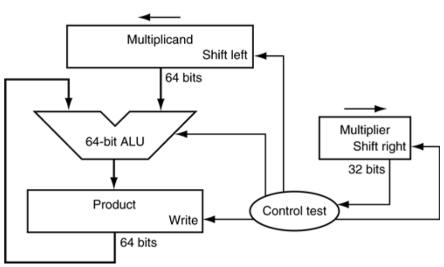
| Operation | Operand A | Operand B | Result Indicating overflow |
|---|---|---|---|
| A+B | ≥0 | ≥0 | <0 |
| A+B | <0 | <0 | ≥0 |
| A-B | ≥0 | <0 | <0 |
| A-B | <0 | ≥0 | ≥0 |

## Multiplication

- Start with long-multiplication approach

```
multiplicand →     1000
multiplier   →   × 1001
                   1000
                  0000
                 0000
                1000
product      →  1001000
```

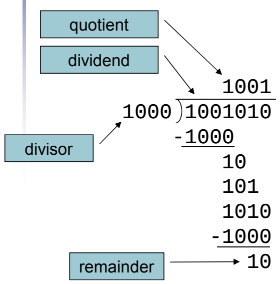
Length of product is the sum of operand lengths




## MIPS Multiplication

- Two 32-bit registers for product
  - HI: most-significant 32 bits
  - LO: least-significant 32-bits
- Instructions
  - `mult rs, rt`
    - 64-bit product in HI/LO
  - `mfhi rd / mflo rd`
    - Move from HI/LO to rd
    - Can test HI value to see if product overflows 32 bits
  - `mul rd, rs, rt`
    - Least-significant 32 bits of product –> rd

## Division




*n*-bit operands yield *n*-bit quotient and remainder

- Check for 0 divisor
- Long division approach
  - If divisor ≤ dividend bits
    - 1 bit in quotient, subtract
  - Otherwise
    - 0 bit in quotient, bring down next dividend bit
- Restoring division
  - Do the subtract, and if remainder goes < 0, add divisor back
- Signed division
  - Divide using absolute values
  - Adjust sign of quotient and remainder as required

## MIPS Division

- Use HI/LO registers for result
  - HI: 32-bit remainder
  - LO: 32-bit quotient
- Instructions
  - `div rs, rt`
  - No overflow or divide-by-0 checking
    - Software must perform checks if required
  - Use `mfhi`, `mflo` to access result

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### Floating Point

## Floating Point

- Representation for non-integral numbers
  - Including very small and very large numbers
- Like scientific notation
  - $-2.34 \times 10^{56}$ ← normalized
  - $+0.002 \times 10^{-4}$ ← not normalized
  - $+987.02 \times 10^{9}$ ← not normalized
- In binary
  - $\pm 1.xxxxxxx_2 \times 2^{yyyy}$
- Types `float` and `double` in C

## Floating Point Standard

- Defined by IEEE Std 754-1985
- Developed in response to divergence of representations
  - Portability issues for scientific code
- Now almost universally adopted
- Two representations
  - Single precision (32-bit)
  - Double precision (64-bit)

## IEEE Floating-Point Format

| S | Exponent<br>single: 8 bits<br>double: 11 bits | Fraction<br>single: 23 bits<br>double: 52 bits |
|---|---|---|

$$x = (-1)^S \times (1 + \text{Fraction}) \times 2^{(\text{Exponent}-\text{Bias})}$$

- S: sign bit (0 ⇒ non-negative, 1 ⇒ negative)
- Normalize significand: 1.0 ≤ |significand| < 2.0
  - Always has a leading pre-binary-point 1 bit, so no need to represent it explicitly (hidden bit)
  - Significand is Fraction with the "1." restored
- Exponent: excess representation: actual exponent + Bias
  - Ensures exponent is unsigned
  - Single: Bias = 127; Double: Bias = 1203

## Single-Precision Range

- Exponents 00000000 and 11111111 reserved
- Smallest value
  - Exponent: 00000001
    ⇒ actual exponent = 1 – 127 = –126
  - Fraction: 000…00 ⇒ significand = 1.0
  - $\pm 1.0 \times 2^{-126} \approx \pm 1.2 \times 10^{-38}$
- Largest value
  - exponent: 11111110
    ⇒ actual exponent = 254 – 127 = +127
  - Fraction: 111…11 ⇒ significand ≈ 2.0
  - $\pm 2.0 \times 2^{+127} \approx \pm 3.4 \times 10^{+38}$

## Double-Precision Range

- Exponents 0000…00 and 1111…11 reserved
- Smallest value
  - Exponent: 00000000001
    ⇒ actual exponent = 1 – 1023 = –1022
  - Fraction: 000…00 ⇒ significand = 1.0
  - $\pm 1.0 \times 2^{-1022} \approx \pm 2.2 \times 10^{-308}$
- Largest value
  - Exponent: 11111111110
    ⇒ actual exponent = 2046 – 1023 = +1023
  - Fraction: 111…11 ⇒ significand ≈ 2.0
  - $\pm 2.0 \times 2^{+1023} \approx \pm 1.8 \times 10^{+308}$

## Floating-Point Precision

- Relative precision
  - all fraction bits are significant
  - Single: approx $2^{-23}$
    - Equivalent to $23 \times \log_{10}2 \approx 23 \times 0.3 \approx 6$ decimal digits of precision
  - Double: approx $2^{-52}$
    - Equivalent to $52 \times \log_{10}2 \approx 52 \times 0.3 \approx 16$ decimal digits of precision

## Activity 1

- Represent $(-0.75)_{10}$ in single and double precision of IEEE 754 binary representation

## Activity 2

- What number is represented by the single-precision float
  11000000101000…00

## Floating-Point Addition

- Consider a 4-digit decimal example
  - $9.999 \times 10^1 + 1.610 \times 10^{-1}$
- 1. Align decimal points
  - Shift number with smaller exponent
  - $9.999 \times 10^1 + 0.016 \times 10^1$
- 2. Add significands
  - $9.999 \times 10^1 + 0.016 \times 10^1 = 10.015 \times 10^1$
- 3. Normalize result & check for over/underflow
  - $1.0015 \times 10^2$
- 4. Round and renormalize if necessary
  - $1.002 \times 10^2$

## Floating-Point Addition

- Now consider a 4-digit binary example
  - $1.000_2 \times 2^{-1} + -1.110_2 \times 2^{-2}$ (0.5 + –0.4375)
- 1. Align binary points
  - Shift number with smaller exponent
  - $1.000_2 \times 2^{-1} + -0.111_2 \times 2^{-1}$
- 2. Add significands
  - $1.000_2 \times 2^{-1} + -0.111_2 \times 2{-1} = 0.001_2 \times 2^{-1}$
- 3. Normalize result & check for over/underflow
  - $1.000_2 \times 2^{-4}$, with no over/underflow
- 4. Round and renormalize if necessary
  - $1.000_2 \times 2^{-4}$ (no change) = 0.0625

## FP Instructions in MIPS

- FP hardware is coprocessor 1
  - Adjunct processor that extends the ISA
- Separate FP registers
  - 32 single-precision: $f0, $f1, … $f31
  - Paired for double-precision: $f0/$f1, $f2/$f3, …
- FP instructions operate only on FP registers
  - Programs generally don't do integer ops on FP data, or vice versa
  - More registers with minimal code-size impact
- FP load and store instructions
  - `lwc1`, `ldc1`, `swc1`, `sdc1`
    - e.g., `ldc1 $f8, 32($sp)`

## FP Instructions in MIPS

- Single-precision arithmetic
  - `add.s`, `sub.s`, `mul.s`, div.s
    - e.g., `add.s $f0, $f1, $f6`
- Double-precision arithmetic
  - `add.d`, `sub.d`, `mul.d`, `div.d`
    - e.g., `mul.d $f4, $f4, $f6`
- Single- and double-precision comparison
  - `c.`*`xx`*`.s`, `c.`*`xx`*`.d` (*xx* is `eq`, `lt`, `le`, …)
  - Sets or clears FP condition-code bit
    - e.g. `c.lt.s $f3, $f4`
- Branch on FP condition code true or false
  - `bc1t`, `bc1f`
    - e.g., `bc1t TargetLabel`

## Concluding Remarks

- ISAs support arithmetic
  - Signed and unsigned integers
  - Floating-point approximation to reals
- Bounded range and precision
  - Operations can overflow and underflow
- MIPS ISA
  - Core instructions: 54 most frequently used
    - 100% of SPECINT, 97% of SPECFP
  - Other instructions: less frequent

## Acknowledgement

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