Instructor Solutions for Assignment 4

Problem 6.1

Solution:

(a) By definition

$$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}\,dt=\int_{-\infty}^{\infty}e^{-5t}u(t)e^{-st}\,dt+\int_{-\infty}^{\infty}e^{4t}u(-t)e^{-st}\,dt=\underbrace{\int_{0}^{\infty}e^{-(s+5)t}\,dt}_{I}+\underbrace{\int_{-\infty}^{0}e^{(4-s)t}\,dt}_{II}.$$

Integral I reduces to

,
$$I=\int_{0}^{\infty}e^{-(s+5)t}\,dt=\left.\frac{e^{-(s+5)t}}{-(s+5)}\right|_{0}^{\infty}=\frac{-1}{(s+5)}\big[0-1\big]=\frac{1}{s+5}\quad \text{provided }\ \mathrm{Re}\{(s+5)\}>0\Rightarrow ROC\ R_1:\mathrm{Re}\{s\}>-5$$

while integral II reduces to

$$II=\int_{-\infty}^{0}e^{(4-s)t}\,dt=\left.\frac{e^{(4-s)t}}{(4-s)}\right|_{-\infty}^{0}=\frac{1}{(4-s)}\big[1-0\big]=\frac{-1}{s-4}\quad \text{provided }\ \mathrm{Re}\{(4-s)\}>0\Rightarrow ROC\ R_1:\mathrm{Re}\{s\}<4\cdot$$

The Laplace transform is therefore given by

$$X(s)=I+II=\frac{1}{s+5}-\frac{1}{s-4}=\frac{-9}{(s+5)(s-4)}\quad \textit{with } ROC: R=R_1\cap R_2\ \textit{or}\ R:(-5<\mathrm{Re}\{s\}<4)\cdot$$

(b) By definition

$$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}\,dt=\int_{-\infty}^{\infty}e^{-3|t|}e^{-st}\,dt=\int_{-\infty}^{0}e^{3t}e^{-st}\,dt+\int_{0}^{\infty}e^{-3t}e^{-st}\,dt=\underbrace{\int_{-\infty}^{0}e^{(3-s)t}\,dt}_{I}+\underbrace{\int_{0}^{\infty}e^{-(s+3)t}\,dt}_{II}.$$

Integral I reduces to

$$I=\int_{-\infty}^{0}e^{(3-s)t}\,dt=\left.\frac{e^{(3-s)t}}{(3-s)}\right|_{-\infty}^{0}=\frac{1}{(3-s)}\big[1-0\big]=\frac{-1}{s-3}\quad \textit{provided}\ \ \mathrm{Re}\{(3-s)\}>0\Rightarrow ROC\ R_1:\mathrm{Re}\{s\}<3$$

while integral II reduces to

$$II=\int_{0}^{\infty}e^{-(s+3)t}\,dt=\left.\frac{e^{-(s+3)t}}{-(s+3)}\right|_{0}^{\infty}=\frac{-1}{(s+3)}\big[0-1\big]=\frac{1}{s+3}\quad \textit{provided}\ \ \mathrm{Re}\{(s+3)\}>0\Rightarrow ROC\ R_1:\mathrm{Re}\{s\}>-3$$

The Laplace transform is therefore given by

$$X(s)=I+II=\frac{1}{s+3}-\frac{1}{s-3}=\frac{-6}{s^2-9}\quad \textit{with } ROC: R=R_1\cap R_2\ \textit{or}\ R:(-3<\mathrm{Re}\{s\}<3)\cdot$$

Problem 6.3

Solution:

(a) Using partial fraction expansion and associating the ROC to individual terms, gives

$$X(s)=\frac{s^2+2s+1}{(s+1)(s^2+5s+6)}=\frac{(s+1)^2}{(s+1)(s+2)(s+3)}=\frac{s+1}{(s+2)(s+3)}=\underbrace{\frac{A}{s+2}}_{\text{ROC:Re}\{s\}>-2}+\underbrace{\frac{B}{s+3}}_{\text{ROC:Re}\{s\}>-3}$$

where $A=\left[\frac{s+1}{s+3}\right]_{s=-2}=-1,\quad B=\left[\frac{s+1}{s+2}\right]_{s=-3}=2$

Taking the inverse transform of $X(s)$, gives

$$x(t)=-e^{-2t}u(t)+2e^{-3t}u(t)=\left(2e^{-3t}-e^{-2t}\right)u(t).$$

(b) Using partial fraction expansion and associating the ROC to individual terms, gives

$$X(s)=\frac{s^2+2s+1}{(s+1)(s^2+5s+6)}=\frac{s+1}{(s+2)(s+3)}=\underbrace{\frac{A}{s+2}}_{\text{ROC:Re}\{s\}<-2}+\underbrace{\frac{B}{s+3}}_{\text{ROC:Re}\{s\}<-3}$$

where constants A, and B were computed in part (a) as $A=-1$, and $B=2$.

Taking the inverse transform of $X(s)$, gives

$$x(t)=\left(e^{-2t}-2e^{-3t}\right)u(-t)$$

Note that the same rational fraction for $X(s)$ gives different time domain representations if the associated ROC is changed.

(e) Using partial fraction expansion and associating the ROC to individual terms, gives

$$X(s)=\frac{s^2+1}{s(s+1)(s^2+2s+17)}=\underbrace{\frac{A}{s}}_{\text{ROC:Re}\{s\}>0}+\underbrace{\frac{B}{(s+1)}}_{\text{ROC:Re}\{s\}>-1}+\underbrace{\frac{Cs+D}{(s^2+2s+17)}}_{\text{ROC:Re}\{s\}>\text{Re}\{-1\pm j4\}}$$

where

$$A=\left[\frac{s^2+1}{s(s+1)(s^2+2s+17)}s\right]_{s=0}=\left[\frac{s^2+1}{(s+1)(s^2+2s+17)}\right]_{s=0}=\frac{1}{17}$$

and $B=\left[\frac{s^2+1}{s(s+1)(s^2+2s+17)}(s+1)\right]_{s=-1}=\left[\frac{s^2+1}{s(s^2+2s+17)}\right]_{s=-1}=-\frac{1}{8}.$

To evaluate C and D, expand $X(s)$ as

$$s^2+1=A(s+1)(s^2+2s+17)+Bs(s^2+2s+17)+(Cs+D)s(s+1)$$

and compare the coefficients of s^3 and s^2. We get

$$0=A+B+C$$
$$1=3A+2B+C+D$$

which has a solution $C = 9/136$ and $D = 137/136$. The Laplace transform may be expressed as

$$X(s)=\underbrace{\frac{1}{17s}}_{\text{ROC:Re}\{s\}>0}-\underbrace{\frac{1}{8(s+1)}}_{\text{ROC:Re}\{s\}>-1}+\underbrace{\frac{9(s+1)}{136((s+1)^2+4^2)}}_{\text{ROC:Re}\{s\}>-1}+\underbrace{\frac{32\times 4}{136((s+1)^2+4^2)}}_{\text{ROC:Re}\{s\}>-1}$$

Taking the inverse transform of $X(s)$, gives

$$x(t)=\tfrac{1}{17}u(t)-\tfrac{1}{8}e^{-t}u(t)+\tfrac{9}{136}e^{-t}\cos(4t)u(t)+\tfrac{4}{17}e^{-t}\sin(4t)u(t).$$

Problem 6.13

Solution:

(a) Calculating the Laplace transform of both sides, we get

$$\left[s^2Y(s)-s\underbrace{y(0^-)}_{=0}-\underbrace{\dot{y}(0^-)}_{=0}\right]+3\left[sY(s)-\underbrace{y(0^-)}_{=0}\right]+2Y(s)=1$$

which reduces to $$(s^2+3s+2)Y(s)=1$$

or, $$Y(s)=\tfrac{1}{(s^2+3s+2)}=\tfrac{1}{(s+1)(s+2)}=\tfrac{1}{s+1}-\tfrac{1}{s+2}.$$

Calculating the inverse Laplace transform, we get

$$y(t)=e^{-t}u(t)-e^{-2t}u(t).$$

(c) Calculating the Laplace transform of both sides, we get

$$\left[s^2Y(s)-s\underbrace{y(0^-)}_{=1}-\underbrace{\dot{y}(0^-)}_{=1}\right]+6\left[sY(s)-\underbrace{y(0^-)}_{=1}\right]+8Y(s)=\tfrac{1}{(s+3)^2}$$

which reduces to $$(s^2+6s+8)Y(s)=\tfrac{1}{(s+3)^2}+(s+1+6)$$

or, $$Y(s)=\tfrac{1}{(s+2)(s+3)^2(s+4)}+\tfrac{s+7}{(s+2)(s+4)}.$$

Taking the partial fraction expansion of the two terms separately

$$\frac{1}{(s+2)(s+3)^2(s+4)}=\frac{1/2}{s+2}+\frac{0}{s+3}-\frac{1}{(s+3)^2}-\frac{1/2}{s+4}$$

and $$\frac{s+7}{(s+2)(s+4)}=\frac{5/2}{s+2}-\frac{3/2}{s+4}$$

Expanding $Y(s)$ as

$$Y(s)=\tfrac{1/2}{s+2}-\tfrac{1}{(s+3)^2}-\tfrac{1/2}{s+4}+\tfrac{5/2}{s+2}-\tfrac{3/2}{s+4}=\tfrac{3}{s+2}-\tfrac{1}{(s+3)^2}-\tfrac{2}{s+4}.$$

Taking the inverse Laplace transform of Y(s) gives

$$y(t)=\left(3e^{-2t}-te^{-3t}-2e^{-4t}\right)u(t)$$

Problem 6.14

(a) The Laplace transform of the input and output signals are given by

$$X(s)=\tfrac{4}{s}\quad\text{and}\quad Y(s)=\tfrac{1}{s^2}+\tfrac{1}{s+2}=\tfrac{s^2+s+2}{s^2(s+2)}.$$

Dividing $Y(s)$ with $X(s)$, the transfer function is given by

$$H(s)=\tfrac{Y(s)}{X(s)}=\tfrac{s^2+s+2}{4s(s+2)}.$$

The impulse response is obtained by taking the partial fraction expansion of $H(s)$ as follows

$$H(s) = \frac{s^2+s+2}{4s(s+2)} \equiv \frac{1}{4} + \frac{1}{4s} - \frac{1}{2(s+2)}.$$

Taking the inverse Laplace transform, the impulse response is given by

$$h(t) = \tfrac{1}{4}\delta(t) + \tfrac{1}{4}u(t) - \tfrac{1}{2}e^{-2t}u(t).$$

In order to calculate the input-output relationship in the form of a differential equation, we represent the transfer function as

$$H(s) = \frac{s^2+s+2}{4s(s+2)} = \frac{Y(s)}{X(s)}.$$

Cross multiplying, we get $4s(s+2)Y(s) = (s^2+s+2)X(s)$

which can be represented as $4s^2Y(s) + 8sY(s) = s^2X(s) + sX(s) + 2X(s)$.

Taking the inverse Laplace transform and assuming zero initial conditions, the differential equation representing the system is given by

$$4\frac{d^2y}{dt^2} + 8\frac{dy}{dt} = \frac{d^2x}{dt^2} + \frac{dx}{dt} + 2x(t).$$

(b) The Laplace transform of the input and output signals are given by

$$X(s) = \frac{1}{(s+2)} \quad \text{and} \quad Y(s) = 3e^{-4s}\frac{1}{(s+2)}.$$

Dividing $Y(s)$ with $X(s)$, the transfer function is given by

$$H(s) = \frac{Y(s)}{X(s)} = 3e^{-4s}.$$

The impulse response is obtained by taking the inverse Laplace transform. The impulse response is given by

$$h(t) = 3\delta(t-4).$$

In order to calculate the input-output relationship in the form of a differential equation, we represent the transfer function as

$$H(s) = 3e^{-4s} = \frac{Y(s)}{X(s)}.$$

Cross multiplying, we get $Y(s) = 3e^{-4s}X(s)$.

Taking the inverse Laplace transform, the input-output relationship of the system is given by

$$y(t) = 3x(t-4).$$

(d) The Laplace transform of the input and output signals are given by

$$X(s) = \frac{1}{s+2} \quad \text{and} \quad Y(s) = \frac{1}{s+1} + \frac{1}{s+3}.$$

Dividing $Y(s)$ with $X(s)$, the transfer function is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{(s+2)}{(s+1)} + \frac{(s+2)}{(s+3)} \equiv 2 + \frac{1}{s+1} - \frac{1}{s+3}.$$

The impulse response is obtained by taking the inverse Laplace transform. The impulse response is given by

$$h(t) = 2u(t) + e^{-t}u(t) - e^{-3t}u(t).$$

In order to calculate the input-output relationship in the form of a differential equation, we represent the transfer function as

$$H(s) = \frac{(s+2)(s+1+s+3)}{(s+1)(s+3)} = \frac{Y(s)}{X(s)}.$$

Cross multiplying, we get $2s^2Y(s) + 8sY(s) + 8Y(s) = s^2X(s) + 4sX(s) + 3X(s)$.

Taking the inverse Laplace transform, the input-output relationship of the system is given by

$$2\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 8y(t) = \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x(t).$$

(e) Note that there is no overlap between the ROC's of the two terms $\exp(t)u(-t)$ and $\exp(-3t)u(t)$, therefore, the Laplace transform for $y(t)$ does not exist.

Problem 6.15

Solution:

(a) $H(s) = \frac{s^2+1}{s^2+2s+1} = \frac{(s+j)(s-j)}{(s+1)^2}$

Two zeros at $s = j, -j$.

Two poles at $s = 1, -1$.

Because both poles are in the left hand side of the s-plane, the system is always BIBO stable.

(b) $H(s) = \frac{2s+5}{s^2+s-6} = 2\frac{(s+2.5)}{(s+3)(s-2)}$

One zero: at $s = -2.5$.

Two poles at $s = 2, -3$.

Because one pole is located in the right hand side of the s-plane, the system is NOT stable.

(c) $H(s) = \frac{3s+10}{s^2+9s+18} = 3\frac{(s+10/3)}{(s+3)(s+6)}$

One zero at $s = -10/3$.

Two poles at $s = -3, -6$.

Because both poles are in the left side of the s-plane, the system is always BIBO stable.

(d) $H(s) = \frac{s+2}{s^2+9} = \frac{(s+2)}{(s+j3)(s-j3)}$

One zero at $s = -2$.

Two poles at $s = j3, -j3$.

There are only two poles and both poles are located on the imaginary axis. Therefore, the system is a marginally stable system.

(e) $H(s) = \frac{s^2+3s+2}{s^3+3s^2+2s} = \frac{1}{s}$

The system does not have any zero.

One pole at $s = 0$.

There is only one pole, which is located on the imaginary axis. Therefore, the system is a marginally stable system.

Problem 6.22

Given the transfer function

$$H(s) = \frac{s^2 - s - 6}{(s^2 + 3s + 1)(s^2 + 7s + 12)}$$

(a) Determine all possible choices for the ROC.

(b) Determine the impulse response of a causal implementation of the transfer function $H(s)$.

(c) Determine the left sided impulse response with the specified transfer function $H(s)$.

(d) Determine all possible choices of double-sided impulse responses having the specified transfer function $H(s)$.

(e) Which of the four impulse responses obtained in (b)-(d) are stable?

Solution:

(a) Factorizing $H(s)$ gives the following expression for the transfer function

$$H(s) = \frac{(s-3)(s+2)}{(s+1)(s+2)(s+3)(s+4)} = \frac{(s-3)}{(s+1)(s+3)(s+4)}.$$

The poles of $H(s)$ are located at $s = -1, -3, -4$. Possible choices of the ROC are:

Choice 1: ROC: $\text{Re}\{s\} > -1$.

Choice 2: ROC: $-3 < \text{Re}\{s\} < -1$.

Choice 3: ROC: $-4 < \text{Re}\{s\} < -3$.

Choice 4: ROC: $\text{Re}\{s\} < -4$.

(b) For a causal implementation of $H(s)$, the ROC must cover most of the right half of the s-plane to ensure that $h_1(t)$ is a right hand sided sequence. The overall ROC is therefore given by ROC: $\text{Re}\{s\} > -1$.

Taking the partial fraction expansion of $H(s)$ gives

$$H(s) = \frac{(s-3)}{(s+1)(s+3)(s+4)} = -\underbrace{\frac{2/3}{(s+1)}}_{\text{ROC:Re}\{s\}>-1} + \underbrace{\frac{3}{(s+3)}}_{\text{ROC:Re}\{s\}>-3} - \underbrace{\frac{7/3}{(s+4)}}_{\text{ROC:Re}\{s\}>-4}.$$

Taking the inverse Laplace transform gives

$$h_1(t) = -\tfrac{2}{3}e^{-t}u(t) + 3e^{-3t}u(t) - \tfrac{7}{3}e^{-4t}u(t).$$

Since all three terms in $h_1(t)$ decay to 0 as $t \to \infty$, $h_1(t)$ is stable.

(c) For a left hand sided implementation of $H(s)$, the ROC must cover most of the left half of the s-plane. The overall ROC is therefore given by ROC: $\text{Re}\{s\} < -4$.

Taking the partial fraction expansion of $H(s)$ gives

$$H(s) = \frac{(s-3)}{(s+1)(s+3)(s+4)} = -\underbrace{\frac{2/3}{(s+1)}}_{\text{ROC:Re}\{s\}<-1} + \underbrace{\frac{3}{(s+3)}}_{\text{ROC:Re}\{s\}<-3} - \underbrace{\frac{7/3}{(s+4)}}_{\text{ROC:Re}\{s\}<-4}.$$

Taking the inverse Laplace transform gives

$$h_2(t) = \tfrac{2}{3}e^{-t}u(-t) - 3e^{-3t}u(-t) + \tfrac{7}{3}e^{-4t}u(-t).$$

Note that $h_2(t)$ is not stable because all three terms are unstable.

(d) For a double sided implementation of $H(s)$, the ROC must consist of a narrow strip within the s-plane. The overall ROC is therefore given by ROC: $(-3 < \text{Re}\{s\} < -1)$, or, ROC: $(-4 < \text{Re}\{s\} < -3)$.

If ROC: $(-3 < \text{Re}\{s\} < -1)$, then $H(s)$ is expressed as

$$H(s) = \frac{(s-3)}{(s+1)(s+3)(s+4)} = -\underbrace{\frac{2/3}{(s+1)}}_{\text{ROC:Re}\{s\}<-1} + \underbrace{\frac{3}{(s+3)}}_{\text{ROC:Re}\{s\}>-3} - \underbrace{\frac{7/3}{(s+4)}}_{\text{ROC:Re}\{s\}>-4}.$$

Taking the inverse Laplace transform gives

$$h_3(t) = \tfrac{2}{3}e^{-t}u(-t) + 3e^{-3t}u(t) - \tfrac{7}{3}e^{-4t}u(t).$$

Note that such $h_3(t)$ is not stable because the term $\tfrac{2}{3}e^{-t}u(-t)$ is not stable.

On the other hand, if ROC: $(-4 < \text{Re}\{s\} < -3)$, then $H(s)$ is expressed as

$$H(s) = \frac{(s-3)}{(s+1)(s+3)(s+4)} = -\underbrace{\frac{2/3}{(s+1)}}_{\text{ROC:Re}\{s\}<-1} + \underbrace{\frac{3}{(s+3)}}_{\text{ROC:Re}\{s\}<-3} - \underbrace{\frac{7/3}{(s+4)}}_{\text{ROC:Re}\{s\}>-4}.$$

Taking the inverse Laplace transform gives

$$h_4(t) = \tfrac{2}{3}e^{-t}u(-t) - 3e^{-3t}u(-t) - \tfrac{7}{3}e^{-4t}u(t).$$

Note that such $h_4(t)$ is not stable because the terms $\tfrac{2}{3}e^{-t}u(-t)$ and $3e^{-3t}u(-t)$ are not stable.

(e) As shown above, the implementation $h_1(t)$ with the overall ROC given by ROC: $\text{Re}\{s\} > -1$ is stable. The remaining implementations $h_2(t)$, $h_3(t)$, and $h_4(t)$ are unstable.