EECS2602.4 Continuous Signals and Systems

No. 4
Laplace Transform

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Laplace Transform: Definition

Analysis equation:
(Inverse Laplace transform)

$$x(t) = \frac{1}{2\pi j} \int_{\sigma+j\infty}^{\sigma-j\infty} X(s) e^{st} ds$$

Synthesis equation:
(Laplace transform)

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

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Laplace vs. CTFT

- CTFT is a special case of Laplace transform.
- Laplace transform is CTFT of a modulated signal.

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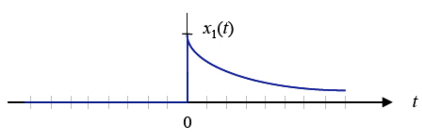
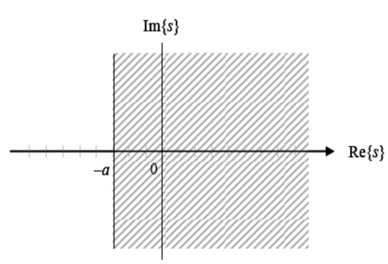
Laplace Transform: example

$$x_1(t) = \exp(-at)u(t),\ a \sim R^+.$$



Solution:

$$X(s) = \begin{cases} \dfrac{1}{(s+a)} & \text{for } \mathrm{Re}\{s\} > -a \\ \text{undefined} & \text{for } \mathrm{Re}\{s\} \le -a. \end{cases}$$



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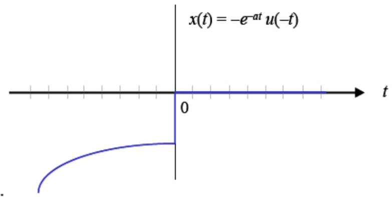
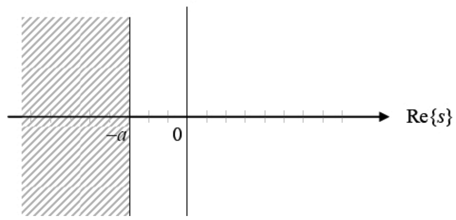
Laplace Transform: example

$x(t) = -\exp(-at)u(-t),\ a \sim R^{+}.$



Solution:

$$X(s) = \begin{cases} \dfrac{1}{(s+a)} & \text{for } \mathrm{Re}\{s\} < -a \\ \text{undefined} & \text{for } \mathrm{Re}\{s\} \geq -a. \end{cases}$$



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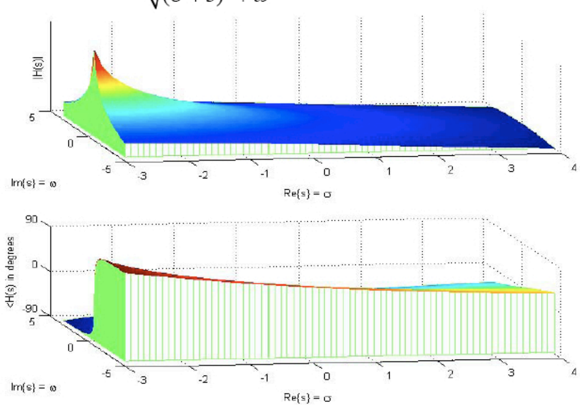
Laplace Spectrum

$x(t) = e^{-3t}\,u(t).$

Solution:

$x(t) = e^{-3t}u(t);$ $\qquad X(s) = \dfrac{1}{s+3}$

$|X(s)| = \dfrac{1}{\sqrt{(\sigma+3)^2+\omega^2}};$ $\qquad < X(s) = -\tan^{-1}\dfrac{\omega}{\sigma+3}$



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Unilateral Laplace Transform

- Laplace transform of causal signals simplifies into unilateral Laplace transform.

$$X(s) = L\{x(t)\} = \int_{0^-}^{\infty} x(t)\mathrm{e}^{-st}\,\mathrm{d}t,$$

Laplace Transform Pairs (I)

$x(t)$	Laplace Transform $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$	Region of Convergence (ROC)
1. $x(t) = \delta(t)$	1	Entire s-plane
2. $x(t) = u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
3. $x(t) = u(t) - u(t-a)$	$\frac{1}{s}\left(1 - e^{-as}\right)$	$\mathrm{Re}\{s\} > 0$
4. $x(t) = e^{-at}\,u(t)$	$\frac{1}{a+s}$	$\mathrm{Re}\{s\} > -a$
5. $x(t) = t\,u(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
6. $x(t) = t^n\,u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$

Laplace Transform Pairs (II)

7. $x(t) = t\, e^{-at}\, u(t)$	$\frac{1}{(a+s)^2}$	$\mathrm{Re}\{s\} > -a$
8. $x(t) = t^n\, e^{-at}\, u(t)$	$\frac{n!}{(a+s)^{n+1}}$	$\mathrm{Re}\{s\} > -a$
9. $x(t) = \cos(\omega_0 t)\, u(t)$	$\frac{s}{\omega_0^2 + s^2}$	$\mathrm{Re}\{s\} > 0$
10. $x(t) = \sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{\omega_0^2 + s^2}$	$\mathrm{Re}\{s\} > 0$
11. $x(t) = \exp(-at)\cos(\omega_0 t)\, u(t)$	$\frac{a+s}{(a+s)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$
12. $x(t) = \exp(-at)\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{(a+s)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$

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Inverse Laplace Transform

- Cauchy theorem: contour integration
- For rational functions
  - Partial fraction expansion
  - Table look-up

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Properties of Laplace Transform (I)

Properties in the time domain	CTFT: $X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	Laplace Transform $X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$
Linearity: $a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$	$a_1X_1(s)+a_2X_2(s)$ ROC : at least $R_1 \cap R_2$
Time Scaling: $x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{\omega}{a})$	$\frac{1}{\lvert a\rvert}X(\frac{s}{a})$ with ROC : aR
Time Shifting: $x(t-t_0)$	$e^{-j\omega_0 t}X(\omega)$	$e^{-st_0}X(s)$ with ROC : R
Frequency / s-domain Shifting: $x(t)e^{j\omega_0 t}$ or $x(t)e^{s_0 t}$	$X(\omega-\omega_0)$	$X(s-s_0)$ with ROC : $R+\mathrm{Re}\{s_0\}$
Time Differentiation: dx/dt	$j\omega X(\omega)$	$sX(s)-x(0^-)$ with ROC : R
Time Integration: $\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(\omega)}{j\omega}+\pi X(0)\delta(\omega)$	$\frac{X(s)}{s}$ with ROC : $R \cap \mathrm{Re}\{s\}>0$,

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Properties of Laplace Transform (II)

Frequency / s-domain Differentiation: $(-t)x(t)$	$-jdX/d\omega$	dX/ds
Duality: $X(t)$	$2\pi x(\omega)$	Not applicable
Time Convolution: $x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$	$X_1(s)X_2(s)$ ROC includes $R_1 \cap R_2$
Frequency / s-domain Convolution: $x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$	$\frac{1}{2\pi}X_1(s)*X_2(s)$ ROC includes $R_1 \cap R_2$
Parsevals Relationship:	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$	Not applicable
Initial value: $x(0^+)$ if it exists	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$	$\lim_{s\to\infty} sX(s)$ provided $s=\infty$ is included in the ROC of $sX(s)$.
Final value: $x(\infty)$ if it exists	Not applicable	$\lim_{s\to 0} sX(s)$ provided $s=0$ is included in the ROC of $sX(s)$.

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Properties of ROC (I)

1. The ROC consists of 2D strips parallel to the j-axis.
2. For a right sided function, the ROC takes the form Re{s} > $\sigma 0$ consisting of the RHS of the s-plane.
3. For a left sided function, the ROC takes the form Re{s} < $\sigma 0$ consisting of the LHS of the s-plane.
4. For a finite duration function, the ROC consists of the entire s-plane except for the possible deletion of the point s = 0.
5. For a double sided function, the ROC takes the form $\sigma 1$ < Re{s} < $\sigma 2$ and is a strip within the s-plane.

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Properties of ROC (II)

6. The ROC of a rational transfer function does not contain any pole.
7. The ROC R for a right sided function with the rational transfer function H(s) is given by R: Re{s} > Re{pr}, where pr is the location of the rightmost pole.
8. The ROC R for a left sided function with the rational transfer function H(s) is given by R: Re{s} > Re{pl} where pl is the leftmost pole.
9. For a stable and causal LTIC system, all poles of the transfer function should lie in the left half s-plane.

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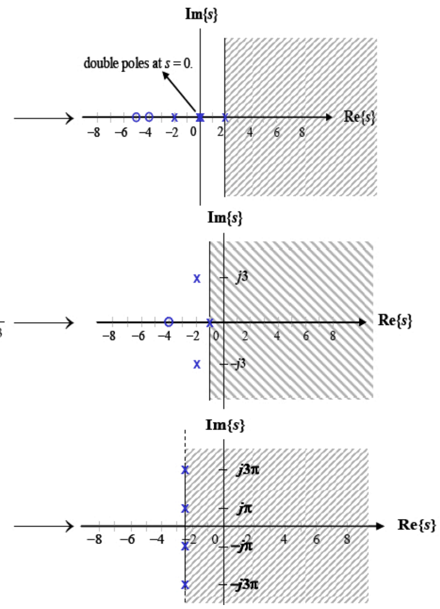
Pole-Zero Plots

$$H_1(s) = \frac{(s+4)(s+5)}{s^2(s+2)(s-2)}$$

$$H_2(s) = \frac{(s+4)}{s^3+5s^2+17s+13}$$

$$H_3(s) = \frac{1}{e^s+10}$$



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Applications of Laplace transform

- Solving Differential Equations
- Laplace analysis of LTIC systems
  - System function (transfer function)
  - Zeros and poles

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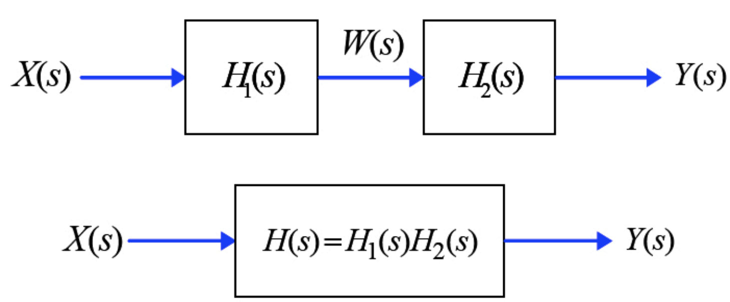
Transfer Function of LTIC Systems

- Transfer function of LTIC is defined as Laplace transform of impulse response.

Laplace Analysis of LTIC systems (I)

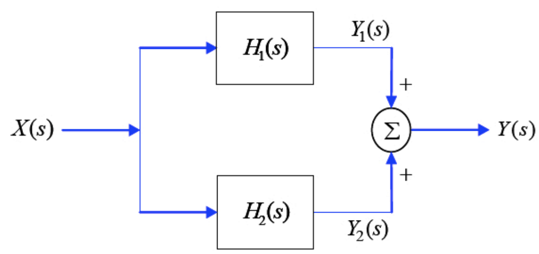
Cascaded Configuration:

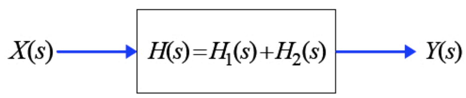


$$h(t) = h_1(t) * h_2(t) \overset{L}{\longleftrightarrow} H(s) = H_1(s)H_2(s)$$

Laplace Analysis of LTIC systems (II)

Parallel Configuration:





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$$h(t) = h_1(t) + h_2(t) \overset{L}{\longleftrightarrow} H(s) = H_1(s) + H_2(s)$$

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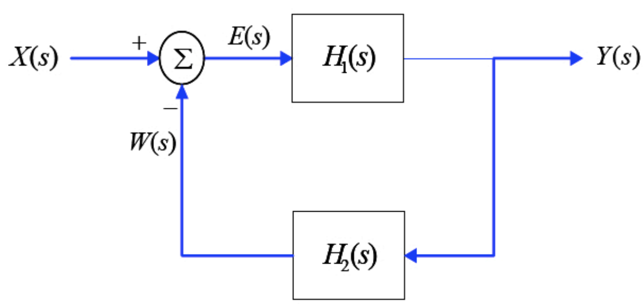
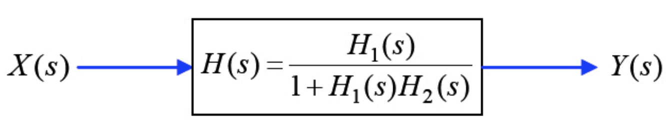
Laplace Analysis of LTIC systems (III)

Feedback Configuration:





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