Concurrent Object Oriented Languages
Linear Temporal Logic

wiki.eecs.yorku.ca/course/6490A
Linear temporal logic (LTL) was proposed by Amir Pnueli to verify computer systems.
Amir Pnueli (1941–2009)

- Recipient of the Turing Award (1996)
- Foreign Associate of the U.S. National Academy of Engineering (1999)
- Fellow of the Association for Computing Machinery (2007)

Source: David Monniaux
\( \varphi ::= \text{true} \mid a \mid \varphi \land \varphi \mid \neg \varphi \mid \Diamond \varphi \mid \varphi \mathbf{U} \varphi \)
Syntax for LTL

\[ \varphi ::= \text{true} \mid a \mid \varphi \land \varphi \mid \neg \varphi \mid \Box \varphi \mid \varphi \mathbf{U} \varphi \]

**Question**

What is the syntactic sugar for \( \varphi_1 \lor \varphi_2 \)?
Syntax for LTL

\[ \varphi ::= \text{true} \mid a \mid \varphi \land \varphi \mid \neg \varphi \mid \Diamond \varphi \mid \varphi \mathbin{U} \varphi \]

Question

What is the syntactic sugar for \( \varphi_1 \lor \varphi_2 \)?

Answer

\( \neg (\neg \varphi_1 \land \neg \varphi_2) \).
Syntactic sugar

\[ \varphi_1 \lor \varphi_2 = \neg (\neg \varphi_1 \land \neg \varphi_2) \]

\[ \varphi_1 \Rightarrow \varphi_2 = \neg \varphi_1 \lor \varphi_2 \]

\[ \Diamond \varphi = \text{true} \lor \varphi \]

\[ \Box \varphi = \neg \Diamond \neg \varphi \]
Alternative syntax

\[ X_{\varphi} : \bigcirc \varphi \]

\[ F_{\varphi} : \lozenge \varphi \]

\[ G_{\varphi} : \Box \varphi \]
Question

How do you express in LTL that “the light is infinitely often red”?
Question
How do you express in LTL that “the light is infinitely often red”?

Answer
□◊red
Question

How do you express in LTL that “once green, the light cannot become red immediately”?
Express properties

Question
How do you express in LTL that “once green, the light cannot become red immediately”?

Answer
□(green ⇒ ¬◯red)
Semantics of LTL

$$TS \models \varphi \iff \forall s \in I : s \models \varphi$$

where

$$s \models \varphi \iff \forall \pi \in Paths(s) : \pi \models \varphi$$

where

$$\begin{array}{ll}
\pi \models true & \\
\pi \models a & \text{iff} \quad a \in L(\pi[0]) \\
\pi \models \varphi \land \psi & \text{iff} \quad \pi \models \varphi \land \pi \models \psi \\
\pi \models \neg \varphi & \text{iff} \quad \pi \not\models \varphi \\
\pi \models \Box \varphi & \text{iff} \quad \pi[1..] \models \varphi \\
\pi \models \varphi U \psi & \text{iff} \quad \exists i \geq 0 : \pi[i..] \models \psi \land \forall 0 \leq j < i : \pi[j..] \models \varphi
\end{array}$$
Semantics of LTL

Question

$\pi \models \text{red}?$
Semantics of LTL

Question

\( \pi \models \text{red?} \)

Answer

Yes.
Semantics of LTL

\[ \pi : 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 5 \rightarrow \ldots \]

Question

\[ \pi \models \mathcal{O} \mathcal{O} \text{red?} \]
Semantics of LTL

Question
\( \pi : 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 5 \rightarrow \cdots \)

\( \pi \models \bigcirc \bigcirc \text{red} ? \)

Answer
No.
Question

\( \pi \models \text{red} \mathbf{U} \text{green}? \)
Question
\(\pi \models \text{red} \mathbb{U} \text{green?}\)

Answer
Yes, because \(L(2) = \{\text{red, amber}\}\).
Semantics of LTL

\[ \pi : \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 5 \rightarrow \ldots \]

**Question**

\[ \pi \models \Diamond \text{black?} \]
Question

$\pi \models \Diamond \text{black}$?

Answer

Yes.
Semantics of LTL

\[ \pi : \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 5 \rightarrow \ldots \]

**Question**

\[ \pi \models \Box \neg \text{red?} \]
Question

\[ \pi \models \square \neg \text{red?} \]

Answer

No.
Semantics of LTL

Question

$\pi \models (\Diamond \text{black}) \mathbin{\mathcal{U}} \Diamond \text{red}$?
Question

$\pi \models (\blacklozenge \text{black}) \cup \lnot \text{red}$?

Answer

Yes.
Definition
The class of LTL formulas $\Box\varphi$, where $\varphi ::= \text{true} \mid a \mid \varphi \land \varphi \mid \neg\varphi$ capture invariants.

Example
$\Box\neg\text{red}$.
Safety properties

The class of LTL formulas that capture that “Nothing bad ever happens” are called safety properties.

Example

“A red light is immediately preceded by amber”

Question

How can we express this property in LTL?
The class of LTL formulas that capture that “Nothing bad ever happens” are called safety properties.

Example
“A red light is immediately preceded by amber”

Question
How can we express this property in LTL?

Answer
□(☐red ⇒ amber).
The class of LTL formulas that capture that “Something good eventually happens” are called *liveness properties*.

**Example**

“the light is infinitely often red”?

**Question**

How can we express this property in LTL?
The class of LTL formulas that capture that “Something good eventually happens” are called *liveness properties*.

**Example**

“the light is infinitely often red”?

**Question**

How can we express this property in LTL?

**Answer**

□♦ red.
Question

Are there properties we cannot express in LTL?
<table>
<thead>
<tr>
<th>Question</th>
<th>Are there properties we cannot express in LTL?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Answer</td>
<td>Yes.</td>
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<tr>
<td>Answer</td>
<td>Yes.</td>
</tr>
<tr>
<td>Example</td>
<td>“Always a state satisfying a can be reached”</td>
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</tbody>
</table>
Expressiveness of LTL

Definition

The predicate $AR_a(TS)$ is defined by

$\forall \pi \in Paths(TS) : \forall m \geq 0 : \exists \pi' \in Paths(\pi[m]) : \exists n \geq 0 : \pi'[n] \models a.$

Theorem

There does not exists an LTL formula $\varphi$ with $TS \models \varphi$ iff $AR_a(TS)$.