EECS2200 Electric Circuits

Chapter 5

Capacitance, Inductance, and RC/RL circuits

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Chapter 5 Part 2

Inductance and RL circuits

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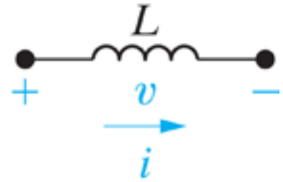
Inductor

Inductor

- Coil of wire with time-varying current
- Inductor equation

$$\boxed{v(t) = L\frac{di(t)}{dt}}$$



Units: v(t) is volts, i(t) is amps, and L is henries [H]

Activity 1

Look at the inductor equation again:

$$v(t) = L\frac{di(t)}{dt}$$

Suppose i(t) is constant, what is v(t)?

A. L
B. 0
C. Undefined

Activity 2

So, if the current in an inductor is constant, its voltage drop is 0, so the inductor can be replaced by:

A. A short circuit.
B. An open circuit.
C. A resistor.

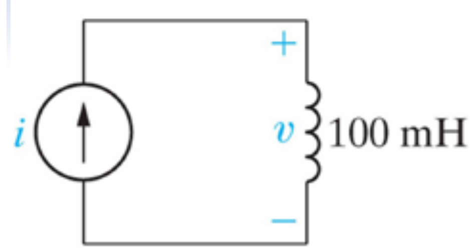
Inductor

- If the current in the inductor is constant, the voltage is 0, so the inductor can be replaced by a SHORT CIRCUIT.
- Look at the inductor equation again: $v(t) = L\frac{di(t)}{dt}$
- Suppose there is a discontinuity in i(t) – that is, at some value of t, the current jumps instantaneously. At this value of t, the derivative of the current is infinite. Therefore the voltage is infinite! NOT POSSIBLE.
- **Thus, the current through an inductor is continuous for all time.**

Activity 3

Given the following conditions,

1. Sketch the current waveform.
2. Find v(t) and sketch voltage waveform.



$$i = 0, \qquad t < 0$$

$$i = 10te^{-5t}\text{A}, \qquad t > 0$$

Current in an Inductor

- The equation for current in terms of voltage:

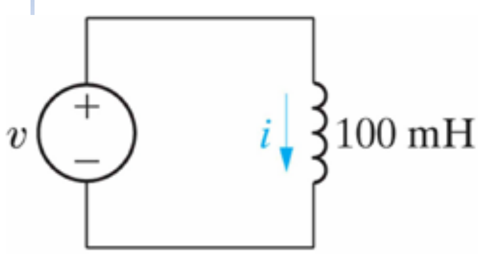
$$v(t) = L\frac{di(t)}{dt} \qquad \Rightarrow \qquad v(t)dt = Ldi(t)$$

$$\Rightarrow \qquad \int_{t_o}^{t} v(\tau)d\tau = L\int_{i(t_o)}^{i(t)} dx$$

$$\Rightarrow \qquad i(t) = \frac{1}{L}\int_{t_o}^{t} v(\tau)d\tau + i(t_o)$$

Activity 4

1. Sketch the voltage as function of time
2. Find i(t)



$$v = 0, \qquad t < 0$$

$$v = 20te^{-10t}\ \text{V}, \qquad t > 0$$

Power and energy

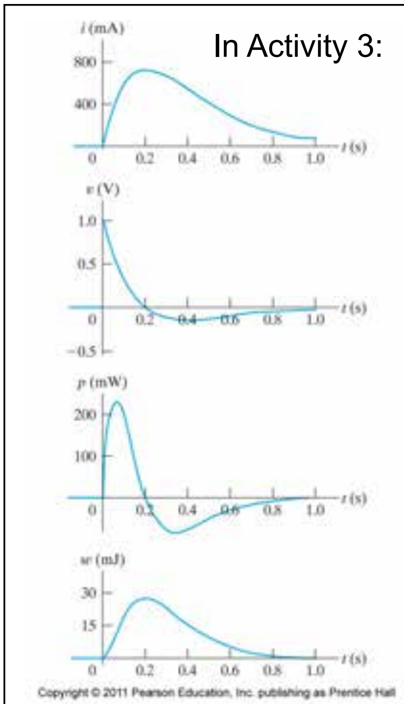
$$p(t) = v(t)i(t) = Li(t)\frac{di(t)}{dt}$$

$$p(t) = \frac{dw(t)}{dt} = Li(t)\frac{di(t)}{dt}$$

$$\Rightarrow \qquad dw(\tau) = Li(\tau)di(\tau)$$

$$\Rightarrow \qquad \int_0^{w(t)} dx = L\int_0^{i(t)} y(\tau)\,d\tau$$

$$\Rightarrow \qquad w(t) = \tfrac{1}{2}Li(t)^2$$



In Activity 3:

$$i(t) = 10te^{-5t}$$

$$v(t) = (1-5t)e^{-5t}$$

$$p(t) = i(t)v(t)$$

$$= 10te^{-10t} - 50t^2e^{-10t}$$

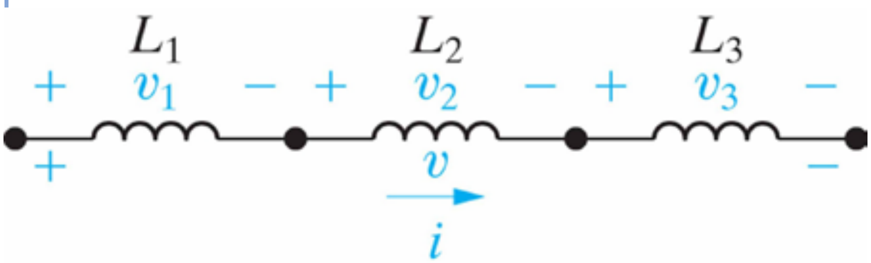
$$\int_0^{0.2} p\,dt = 10\left[\frac{e^{-10t}}{100}(-10t-1)\right]_0^{0.2}$$

$$= 27.07mJ$$

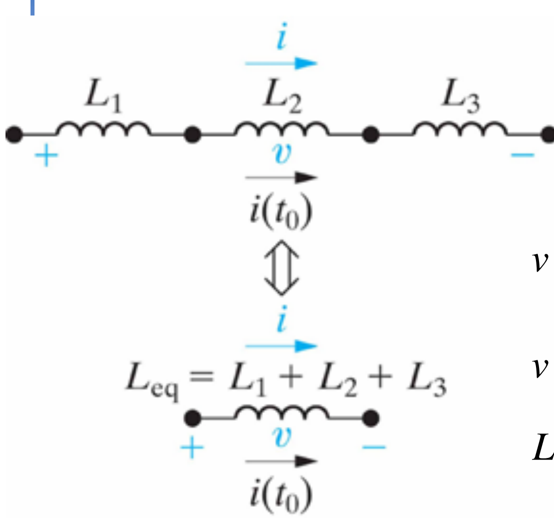
$$\int_{0.2}^{\infty} p\,dt = 10\left[\frac{e^{-10t}}{100}(-10t-1)\right]_{0.2}^{\infty}$$

$$= -27.07mJ$$

Inductors in series



Inductors in series

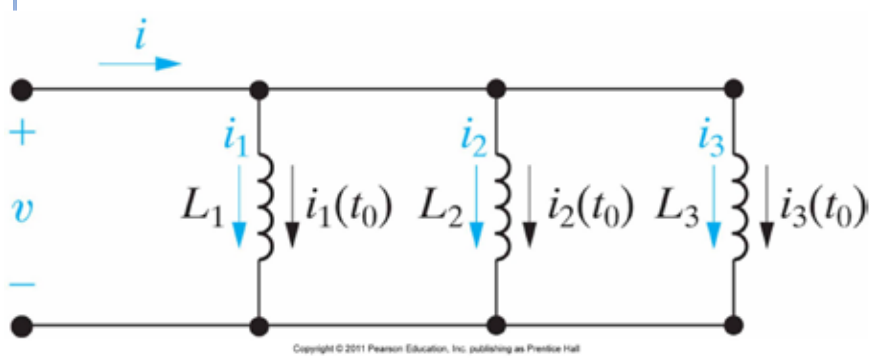


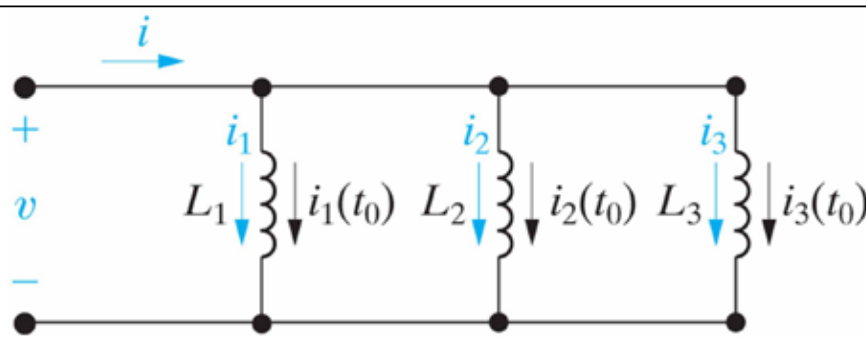
$$v = L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt}$$

$$v = \left(L_1 + L_2 + L_3\right)\frac{di}{dt}$$

$$L_{eq} = L_1 + L_2 + L_3$$

Inductors in parallel

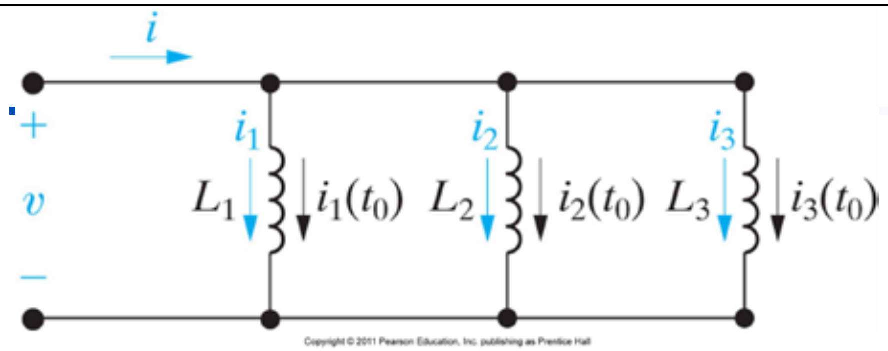


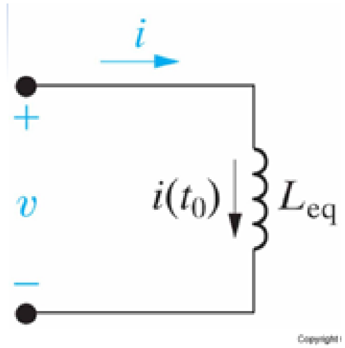


$$i = i_1 + i_2 + i_3$$

$$i(t) = \frac{1}{L_1}\int_0^t v\,d\tau + i_1(t_0) + \frac{1}{L_2}\int_0^t v\,d\tau + i_2(t_0) + \frac{1}{L_3}\int_0^t v\,d\tau + i_3(t_0)$$

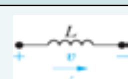
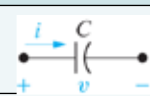
$$i(t) = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)\int_0^t v\,d\tau + i_1(t_0) + i_2(t_0) + i_3(t_0)$$





$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

Inductor and Capacitor comparison

	Inductor	Capacitor
Symbol		
Units	Henries [H]	Farads [F]
Describing equation	$v(t) = L\frac{di(t)}{dt}$	$i(t) = C\frac{dv(t)}{dt}$
Other equation	$i(t) = \frac{1}{L}\int_{t_o}^{t} v(\tau)d\tau + i(t_o)$	$v(t) = \frac{1}{C}\int_{t_o}^{t} i(\tau)d\tau + v(t_o)$
Initial condition	$i(t_o)$	$v(t_o)$
Behavior with const. source	If $i(t) = I$, $v(t) = 0$ → short circuit	If $v(t) = V$, $i(t) = 0$ → open circuit
Continuity requirement	$i(t)$ is continuous so $v(t)$ is finite	$v(t)$ is continuous so $i(t)$ is finite

Inductor and Capacitor comparison

	Inductor	Capacitor
Power	$p(t) = v(t)i(t) = Li(t)\frac{di(t)}{dt}$	$p(t) = v(t)i(t) = Cv(t)\frac{dv(t)}{dt}$
Energy	$w(t) = \frac{1}{2}Li(t)^2$	$w(t) = \frac{1}{2}Cv(t)^2$
Initial energy	$w_o(t) = \frac{1}{2}Li(t_o)^2$	$w_o(t) = \frac{1}{2}Cv(t_o)^2$
Trapped energy	$w(\infty) = \frac{1}{2}Li(\infty)^2$	$w(\infty) = \frac{1}{2}Cv(\infty)^2$
Series-connected	$L_{eq} = L_1 + L_2 + L_2$ $i_{eq}(t_o) = i(t_o)$	$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$ $v_{eq}(t_o) = v_1(t_o) + v_2(t_o) + v_3(t_o)$
Parallel-connected	$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$ $i_{eq}(t_o) = i_1(t_o) + i_2(t_o) + i_3(t_o)$	$C_{eq} = C_1 + C_2 + C_2$ $v_{eq}(t_o) = v(t_o)$

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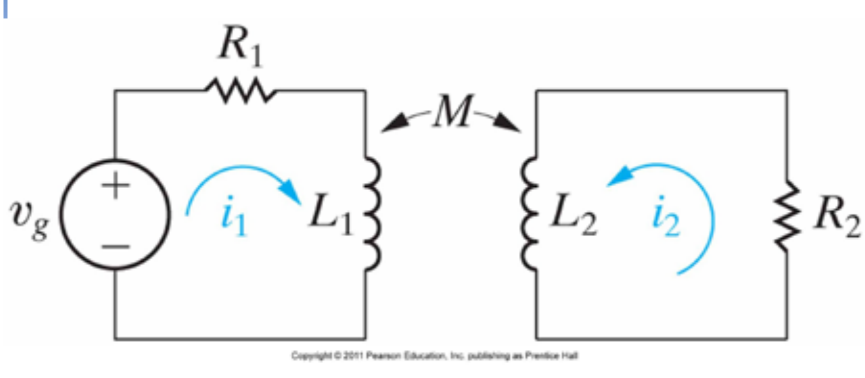
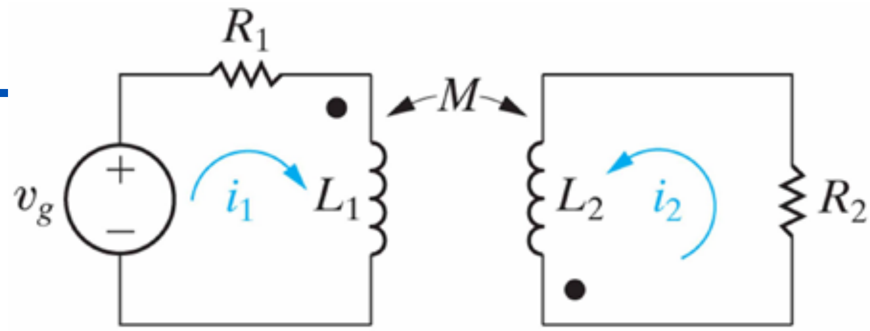
Mutual Inductance

Mutual Inductance

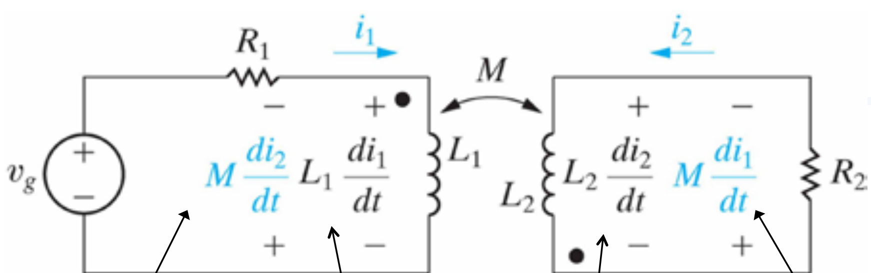
- So far, we are studying a single circuit element, where the change in the current affects the voltage across that element (and vise versa).
- Now, we consider two circuits linked by a magnetic field, such that changes in the current in the first circuit affect the voltage in the second circuit.

Mutual Inductance





Dot Convention: When the reference direction for a current entering (leaving) the dotted terminal in a coil, the reference polarity of the voltage that it induces in the other coil is positive (negative) at its dotted terminal.

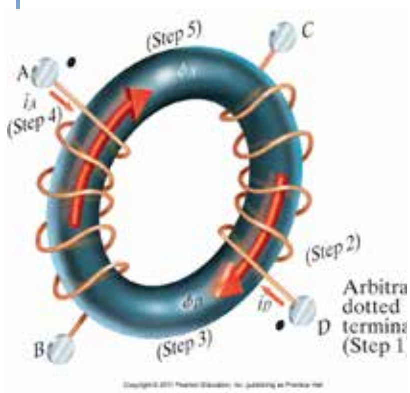


Induced by i_2 Self Self Induced by i_1

$$-v_g + i_1 R_1 + L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = 0$$

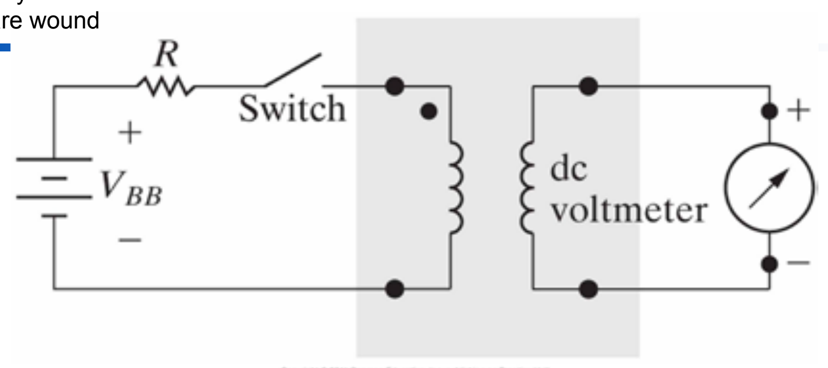
$$i_2 R_2 + L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} = 0$$

How to determine Polarity
1- you know how coils are wound



1. Arbitrarily mark one terminal of one coil (D)
2. Assign a current into that terminal (i_D)
3. Use the right-hand rule to determine the direction of the magnetic flux inside the coils
4. Arbitrarily pick one terminal (A) of the other coil and assign a current into it (i_A)
5. Use the right-hand rule to determine the direction of the magnetic flux from this current inside the coils
6. Compare the two fluxes directions: If the direction is the same place a dot at the referenced terminal (A) if they are opposite select the opposite terminal (B).

How to determine Polarity
2- you do not know how coils are wound

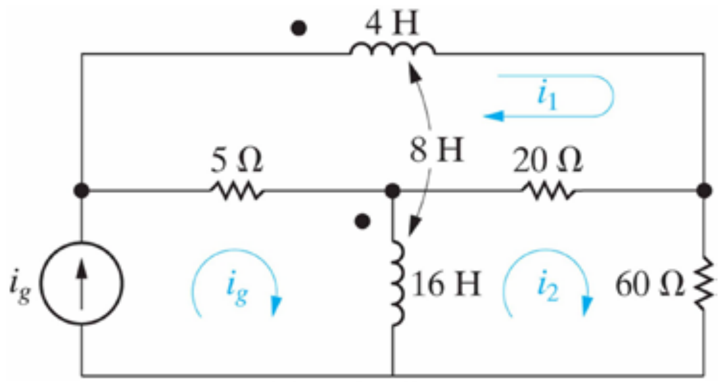


When the switch is closed…

- The second polarity mark goes on the terminal connected to the **positive terminal** of the voltmeter if the voltmeter momentarily deflects **upscale**
- The second polarity mark goes on the terminal connected to the **negative terminal** of the voltmeter if the voltmeter momentarily deflects **downscale**

Activity 5

Write a set of mesh-current equations.



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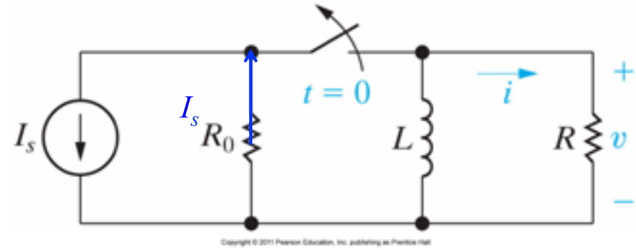
RL Natural Response

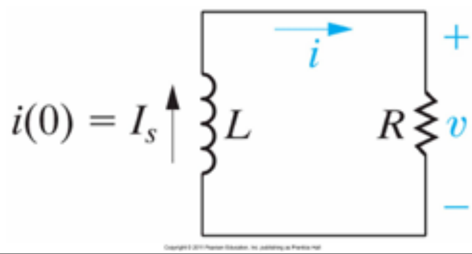
RL Natural Response



The switch is ON for a long time. L is a short circuit, no voltage across R (no current in R).

All of I_s goes directly to L

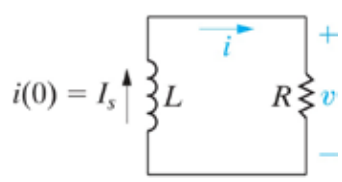
Then the switch is turned OFF (open)



RL Natural Response

Apply KVL:

$$L\frac{di(t)}{dt} + Ri(t) = 0$$



This equation is a

- •First-order (the highest derivative is a first derivative)
- •Homogeneous (the right-hand side is 0)
- •Ordinary differential equation
- •With constant coefficients

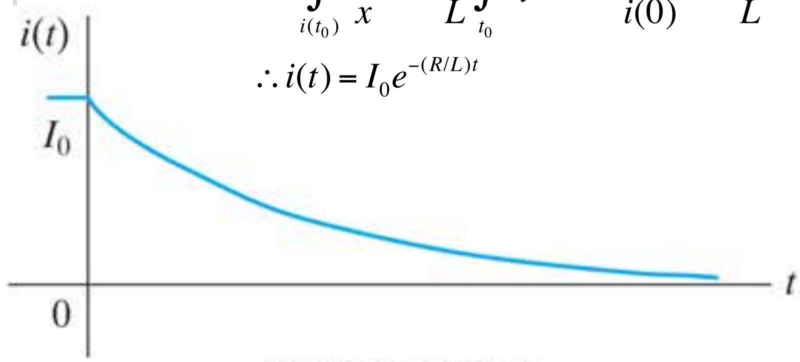
RL Natural Response

$v(0^-) = 0, v(0^+) = I_0 R, \quad i(0^-) = i(0^+) = I_0$

$$L\frac{di}{dt} + Ri = 0 \Rightarrow \frac{di}{dt}dt = -\frac{R}{L}idt \Rightarrow \frac{di}{i} = -\frac{R}{L}dt$$

$$\Rightarrow \int_{i(t_0)}^{i(t)} \frac{dx}{x} = -\frac{R}{L}\int_{t_0}^{t} dy \Rightarrow \ln\frac{i(t)}{i(0)} = -\frac{R}{L}t$$

$$\therefore i(t) = I_0 e^{-(R/L)t}$$



Power and Energy

$$p = vi = i^2 R = I_0^2 R e^{-2(R/L)t}$$

$$w = \int_0^t p(x)\,dx = \int_0^t I_0^2 R e^{-2(R/L)x}\,dx$$

$$= \frac{-1}{2R/L} I_0^2 R \quad e^{-2(R/L)x}\Big|_0^t$$

$$w = \frac{1}{2} L I_0^2 (1 - e^{-2(R/L)t})$$

Time Constant

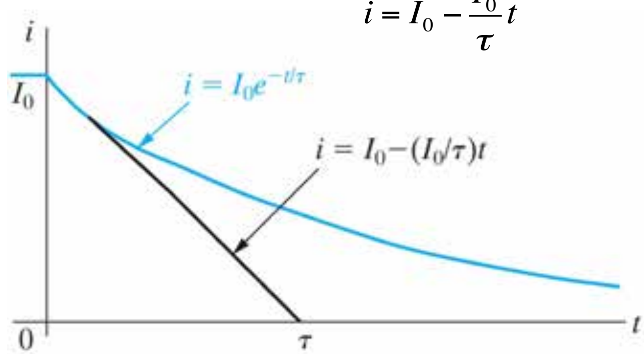
$$i(t) = I_0 e^{-(R/L)t} = I_0 e^{-t/\tau}, \quad \tau = \frac{L}{R} \text{ time constant}$$

- R/L determines the rate at which the current or voltage approaches zero.
- τ (=L/R) is called time constant.
- 1 time constant after the inductor has begun to release its stored energy to the resistor, the current has been reduced to approximately 0.37 (e^{-1}) of its initial value.
- 5 time constant → the current is less than 1% (e^{-5}) of its initial value.

Time Constant

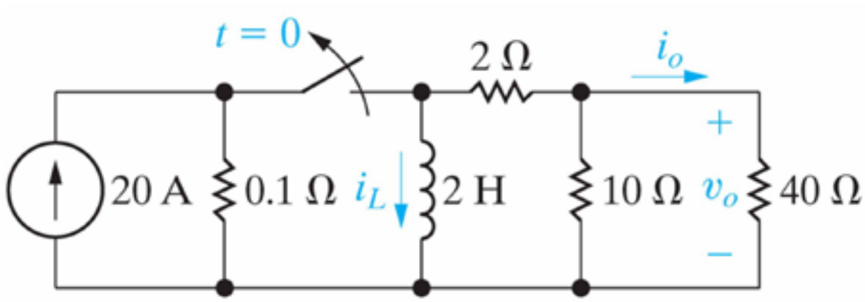
- Estimate time constant from RL natural response.
  - Evaluate di/dt at 0^+: $\frac{di}{dt}(0^+) = -\frac{R}{L} I_0 = -\frac{I_0}{\tau}$
  - If i starts as I_0 and decreases at constant rate I_0/τ

$$i = I_0 - \frac{I_0}{\tau} t$$



Activity 6

The switch has been closed for long time before it is opened at t=0. Find $i_L(t)$ for >=0.



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A general solution method for RL/RC natural response

RL/RC Natural Response

1. Identify the variable of interest (hint – it's the variable that must be continuous in the circuit):
   - For RL, *i(t)* through L
   - For RC, *v(t)* across C
2. Find the initial value of this variable, either $i(0) = I_o$ or $v(0) = V_o$
   - From the problem statement.
   - By analyzing the circuit for $t < 0$, with L replaced by a short circuit or C replaced by an open circuit

RL/RC Natural Response

3. Find the time constant, τ
   - $\tau_{RL} = L/R_{eq}$ or $\tau_{RC} = R_{eq}C$
   - Note that R_{eq} is the equivalent resistance as seen from the inductor or capacitor for $t \geq 0$
4. Write the expression for the variable of interest:

   $$x(t) = X_o e^{-t/\tau}, \quad t \geq 0$$

5. Use simple circuit analysis to calculate any other requested variables.

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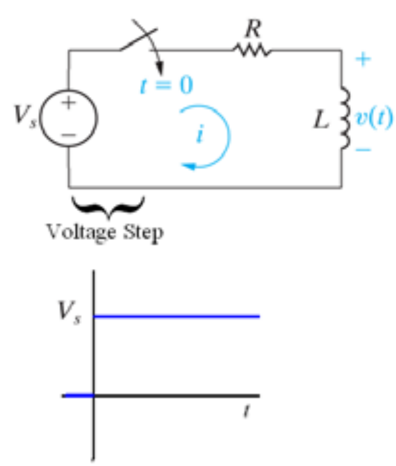
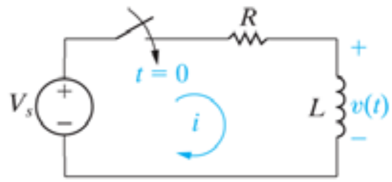
RL Step Response

Step Response of RL Circuits

- This is the step response of this circuit because for $t \geq 0$ there is an independent source in the circuit – the response is due to the initial energy stored in the inductor as well as the energy supplied by the voltage source for $t \geq 0$.



RL Step Response



KVL for $t \geq 0$:

$$-V_S + L\frac{di(t)}{dt} + Ri(t) = 0$$

$$\Rightarrow \quad L\frac{di(t)}{dt} + Ri(t) = V_S$$

This equation is a
- First-order (the highest derivative is a first derivative)
- Inhomogeneous (the right-hand side is non-zero)
- Ordinary differential equation
- With constant coefficients

The left-hand side of the equation is identical to the describing differential equation for the RL Natural Response!

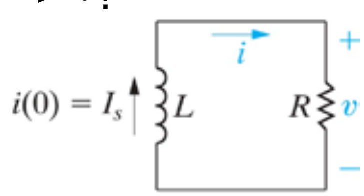
Activity 7

The circuit we analyzed for the RL natural response problem is shown below, along with the expression for *i(t)*, for $t \geq 0$. What is the value of *i(t)* as $t \to \infty$?

A. *0*

B. I_S

C. ∞



$$i(t) = I_s e^{-(R/L)t}, \qquad t \geq 0$$

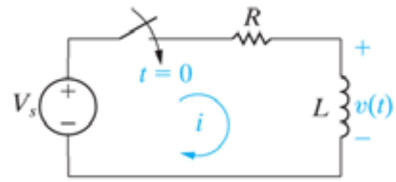
Activity 8

Now consider what happens to the current as $t \rightarrow \infty$ in the RL step response circuit. First, as $t \rightarrow \infty$, we can replace the inductor by

A. An open circuit

B. A short circuit

C. A resistor



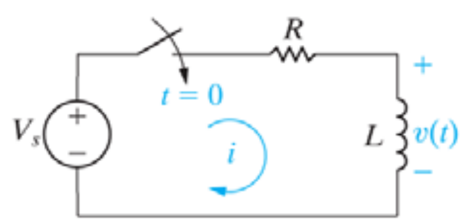
Activity 9

To evaluate the current in the inductor as $t \rightarrow \infty$, close the switch and replace the inductor with a short circuit. The current in the short circuit is

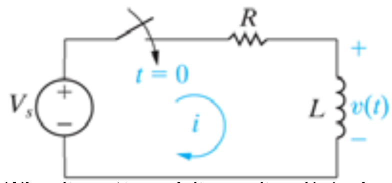
A. 0

B. infinite

C. V_S /R



RL Step Response



KVL : $L\dfrac{di(t)}{dt} + Ri(t) = V_S$

The functional form for $i(t)$ should be the same as for the natural response.

Try $\quad i(t) = I_0 e^{-(R/L)t}$, where I_0 is the initial inductor current.

But as $t \to \infty \quad i(\infty) = I_0 e^{-(R/L)(\infty)} = 0 \quad$ which is incorrect!

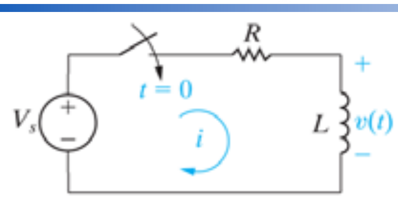
Try $\quad i(t) = I_F + I_0 e^{-(R/L)t}$,

where I_F is the inductor current as $t \to \infty$.

Then $\quad i(\infty) = I_F + I_0 e^{-(R/L)(\infty)} = I_F \quad$ (OK)

But $\quad i(0) = I_F + I_0 e^{-(R/L)(0)} = I_F + I_0 \quad$ (NO!)

RL Step Response



KVL : $L\dfrac{di(t)}{dt} + Ri(t) = V_S$

Now try $\quad i(t) = I_F + (I_0 - I_F)e^{-(R/L)t}$

Then $\quad i(\infty) = I_F + (I_0 - I_F)e^{-(R/L)(\infty)} = I_F \quad$ (OK)

And $\quad i(0) = I_F + (I_0 - I_F)e^{-(R/L)(0)} = I_F + I_0 - I_F = I_0 \quad$ (OK)

Make sure that this $i(t)$ satisfies the differential equation:

$i(t) = I_F + (I_0 - I_F)e^{-(R/L)t}; \qquad \dfrac{di(t)}{dt} = -(R/L)(I_0 - I_F)e^{-(R/L)t}$

Substituting into KVL,

$\mathrm{L}[-(R/L)(I_0 - I_F)e^{-(R/L)t}] + R[I_F + (I_0 - I_F)e^{-(R/L)t}] = RI_F = V_S$

RL Step Response

$$i(t) = I_F + (I_0 - I_F)e^{-(R/L)t}, I_F = \frac{V_s}{R},$$

If initial enery in the inductor is zero, $I_0 = 0$

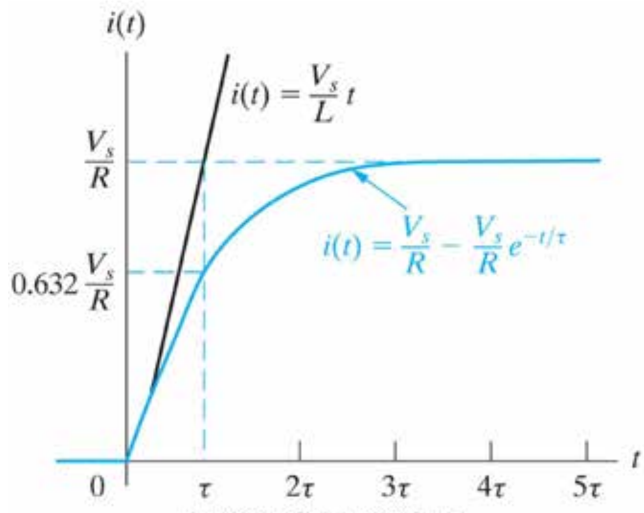
$$\therefore i(t) = \frac{V_s}{R} - \frac{V_s}{R}e^{-(R/L)t}$$

$$v(t) = L\frac{di(t)}{dt} = L\left(-\frac{R}{L}\right)(I_0 - I_F)e^{-(R/L)t}$$

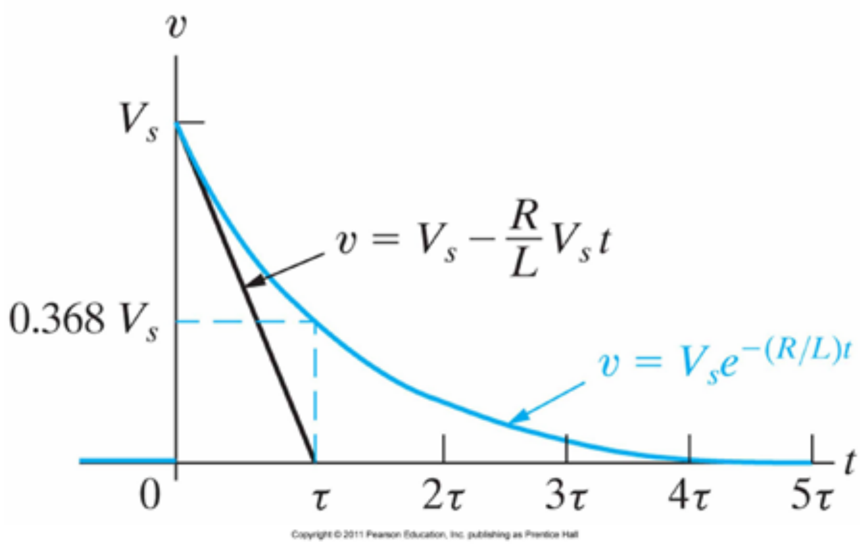
If initial enery in the inductor is zero, $I_0 = 0$

$$\therefore v(t) = V_s e^{-(R/L)t}$$

RL Step Response



RL Step Response



EECS2200 Electric Circuits

A general method for step response of RL/RC circuits

RL/RC Step (Natural) Response

1. Identify the variable of interest (hint – it's the variable that must be continuous in the circuit):
   - For RL, *i(t)* through L
   - For RC, *v(t)* across C
2. Find the initial value of this variable, either $i(0) = I_o$ or $v(0) = V_o$
   - From the problem statement
   - By analyzing the circuit for $t < 0$, with L replaced by a short circuit or C replaced by an open circuit

RL/RC Step (Natural) Response

3. Find the final value of this variable, either $i(\infty) = I_F$ or $v(\infty) = V_F$
   - If there is no source in the circuit as $t \to \infty$, the final value is 0 – this is the natural response problem!
   - Otherwise, analyze the circuit as $t \to \infty$, with L replaced by a short circuit or C replaced by an open circuit
4. Find the time constant, τ
   - $\tau_{RL} = L/R_{eq}$ or $\tau_{RC} = R_{eq}C$
   - Note that R_{eq} is the equivalent resistance as seen from the inductor or capacitor for $t \geq 0$

RL/RC Step (Natural) Response

5. Write the expression for the variable of interest:

   $x(t) = X_F + (X_o - X_F)e^{-t/\tau}, \quad t \geq 0$

6. Use simple circuit analysis to calculate any other requested variables.

Activity 10

1. Find i(t) for t>=0
2. What is voltage across inductor just after switch has been moved to b.
3. How many ms after the switch has been moved does the inductor voltage equal 24V?