Linear Temporal Logic

EECS 4315

www.cse.yorku.ca/course/4315/

Linear temporal logic (LTL) is a logic to reason about systems with nondeterminism.

The logic was introduced by Amir Pnueli.

A. Pnueli. The temporal logic of programs. In *Proceedings of the 18th IEEE Symposium on Foundations of Computer Science*, pages 46–67. Providence, RI, USA, October/November 1977. IEEE.

Amir Pnueli (1941–2009)

- Recipient of the Turing Award (1996)
- Recipient of the Israel prize (2000)
- Foreign Associate of the U.S. National Academy of Engineering (1999)
- Fellow of the Association for Computing Machinery (2007)

Source: David Monniaux

Definition

LTL is defined by the following grammar.

$$\varphi ::= \text{true} \mid a \mid \varphi \wedge \varphi \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi \,\mathsf{U}\, \varphi$$

where a is an atomic proposition.

An atomic proposition represents a basic property (such as the value of a particular variable being even).

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true is always satisfied.

LTL

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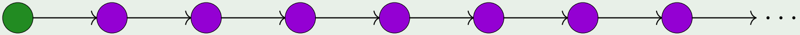
Question

Does the execution path



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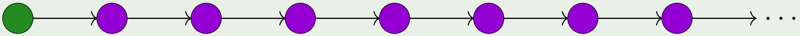

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Answer

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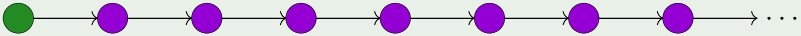

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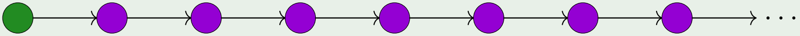

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Answer

No.

LTL

Given an execution path π, does it satisfy a particular LTL formula φ?

The LTL formula $\bigcirc a$ (pronounced as next *a*) is satisfied if *a* holds in the next state of the execution path (that is, the second state).

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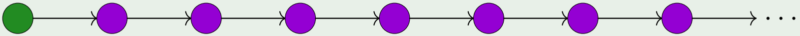

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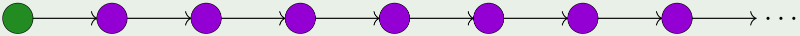

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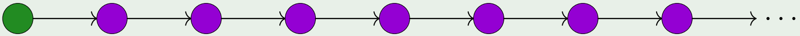

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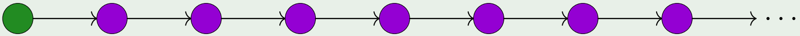

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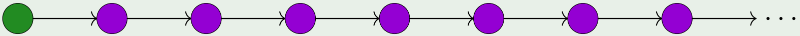

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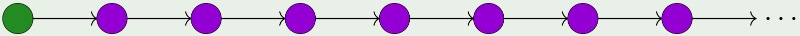

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Given an execution path π, does it satisfy a particular LTL formula φ?

The LTL formula $a \, \mathrm{U} \, b$ (pronounced as *a* until *b*) is satisfied if *b* holds in some state of the execution path and *a* holds in all states before that state.

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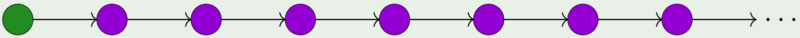

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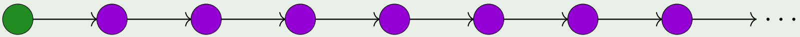

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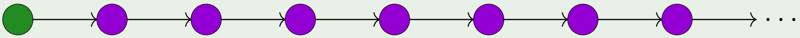

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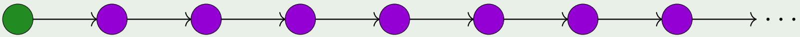

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Answer

Yes.

Syntactic sugar

As usual

$$\begin{aligned} \varphi_1 \vee \varphi_2 &= \neg(\neg\varphi_1 \wedge \neg\varphi_2) \\ \varphi_1 \Rightarrow \varphi_2 &= \neg\varphi_1 \vee \varphi_2 \end{aligned}$$

Also

$$\begin{aligned} \Diamond\varphi &= \text{true U } \varphi \quad &\text{(eventually } \varphi\text{)} \\ \Box\varphi &= \neg\Diamond\neg\varphi \quad &\text{(always } \varphi\text{)} \end{aligned}$$

Alternative syntax

$$\begin{array}{rcl} \mathsf{X}\varphi & : & \bigcirc\varphi \\ \mathsf{F}\varphi & : & \Diamond\varphi \\ \mathsf{G}\varphi & : & \Box\varphi \end{array}$$

State Space Diagram

Question

Draw the state space diagram of a model of a traffic light.

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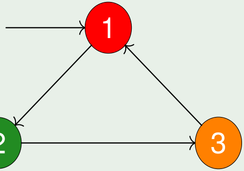
Answer



Note: the transitions are not labelled, but the states are labelled.

Transition System

Definition

A transition system is a tuple $\langle S, L, I, \rightarrow, \ell \rangle$ consisting of

- a set S of states,
- a set L of labels,
- A set $I \subseteq S$ of initial states,
- a transition relation $\rightarrow \subseteq S \times S$ such that for all $s \in S$ there exists a $s' \in S$ such that $s \rightarrow s'$, and
- a labelling function $\ell : S \rightarrow 2^L$.

2^L denotes the set of subsets of L.

Powerset

Question

What is $2^{\{1,2,3\}}$?

Powerset

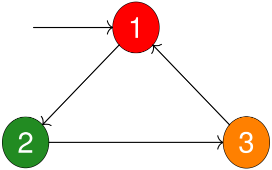
Question

What is $2^{\{1,2,3\}}$?

Answer

$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

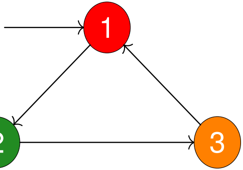
Transition System



Question

Give the transition system modelling a traffic light.

Transition System

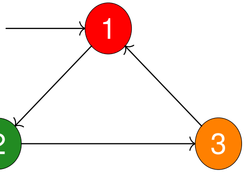


Question

Give the transition system modelling a traffic light.

Answer

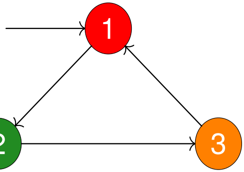
$$\langle \{1, 2, 3\}, \{\text{red}, \text{green}, \text{orange}\},$$
$$\{1\}, \{1 \to 2, 2 \to 3, 3 \to 1\}$$
$$\{1 \mapsto \{\text{red}\}, 2 \mapsto \{\text{green}\}, 3 \mapsto \{\text{orange}\}\}\rangle$$



$$\varphi ::= \text{true} \mid a \mid \varphi \wedge \varphi \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi \,\mathbf{U}\, \varphi$$

Question

Which LTL formula expresses
"initially the light is red."



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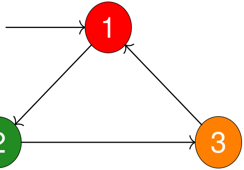
The LTL formula red.



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Which LTL formula expresses

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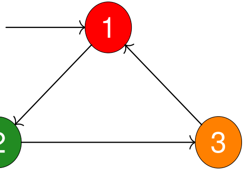

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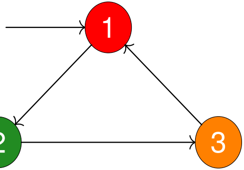
LTL



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Which LTL formula expresses
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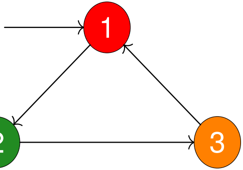

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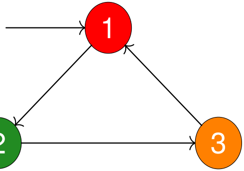
Answer

The LTL formula true U orange $= \Diamond$orange



$$\varphi ::= \text{true} \mid a \mid \varphi \wedge \varphi \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi \,\mathsf{U}\, \varphi$$

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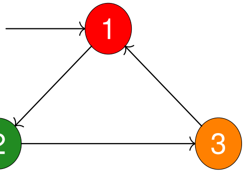
Which LTL formula expresses

"the light is infinitely often red."



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Which LTL formula expresses
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Answer

The LTL formula $\Box\Diamond$red

Question

What does the LTL formula $\Box(\text{green} \Rightarrow \neg \bigcirc \text{red})$ express?

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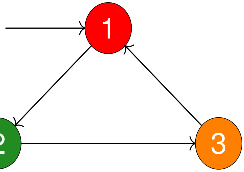
Answer

"Once green, the light cannot become red immediately"

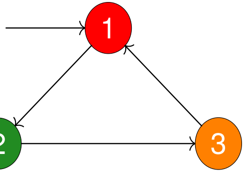


Definition

Paths(s) is the set of path starting in state s.

Question

What is *Paths*(2)?



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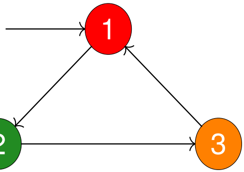
$\{231231231231\cdots\}$

Execution Paths



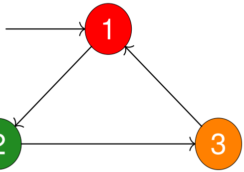
Definition

Let $\pi \in Paths(s)$ and $n \geq 0$. Then $\pi[n]$ is the $(n+1)^{\text{th}}$ state of the path π.

Question

Let $\pi = 123123\cdots$. What is $\pi[3]$?

Execution Paths



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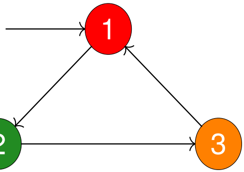

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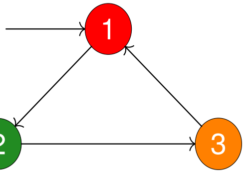

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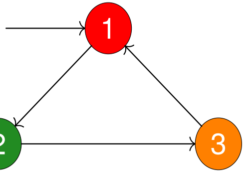
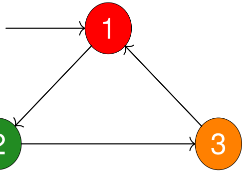


$\pi \models \varphi$ denotes that path π satisfies LTL formula φ

Semantics of LTL



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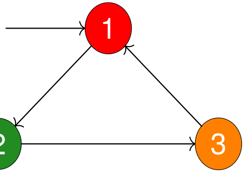

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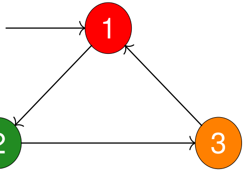

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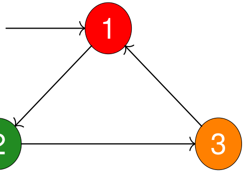

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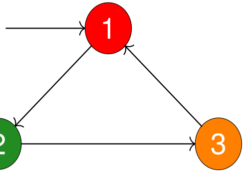

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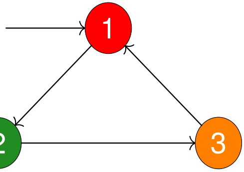
$123123\cdots \models$ red $\wedge \bigcirc$green?



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$123123\cdots \models \text{red} \wedge \bigcirc \text{green}$?

Answer

Yes.

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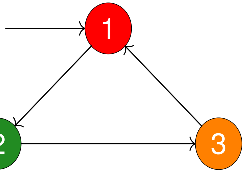

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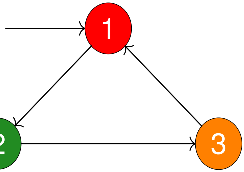

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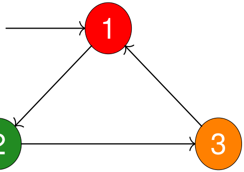

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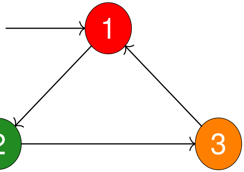
Question

$123123\cdots \models$ red U green?



$\pi \models \varphi$ denotes that path π satisfies LTL formula φ

Question

$123123\cdots \models$ red U green?

Answer

Yes.

Semantics of LTL

Definition

$$
\begin{array}{rcl}
\pi \models \text{true} & & \\
\pi \models a & \text{iff} & a \in \ell(\pi[0]) \\
\pi \models \varphi \wedge \psi & \text{iff} & \pi \models \varphi \wedge \pi \models \psi \\
\pi \models \neg\varphi & \text{iff} & \pi \not\models \varphi \\
\pi \models \bigcirc\varphi & \text{iff} & \pi[1..] \models \varphi \\
\pi \models \varphi \,\mathsf{U}\, \psi & \text{iff} & \exists i \geq 0 : \pi[i..] \models \psi \wedge \forall 0 \leq j < i : \pi[j..] \models \varphi
\end{array}
$$