Linear Temporal Logic

EECS 4315

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Weak Until

Definition

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$$\varphi \, \mathsf{W} \, \psi = (\varphi \, \mathsf{U} \, \psi) \lor \Box \phi$$

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Question

$\pi \models \varphi \, \mathsf{W} \, \psi$ iff?

Answer

$$(\exists i \geq 0 : \pi[i..] \models \psi \text{and} \forall 0 \leq j < i : \pi[j..] \models \varphi) \text{or} \forall i \geq 0 : \pi[i..] \models \varphi$$

Positive Normal Form

Definition

The set of LTL formulas in positive normal form is defined by the grammar

$$\varphi ::= \text{true} \mid \text{false} \mid a \mid \neg a \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \bigcirc \varphi \mid \varphi \,\mathsf{U}\, \varphi \mid \varphi \,\mathsf{W}\, \varphi$$

Positive Normal Form

Definition

The set of LTL formulas in positive normal form is defined by the grammar

$$\varphi ::= \text{true} \mid \text{false} \mid a \mid \neg a \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \bigcirc \varphi \mid \varphi \, \mathsf{U} \, \varphi \mid \varphi \, \mathsf{W} \, \varphi$$

Theorem

For each LTL formula there exists an equivalent LTL formula in positive normal form.

Expressiveness of LTL

Question

Are there properties we cannot express in LTL?

Expressiveness of LTL

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Are there properties we cannot express in LTL?

Answer

Yes, for example, “Always a state satisfying a can be reached”

Expressiveness of LTL

Theorem

There does not exists an LTL formula φ *with* $TS \models \varphi$ *iff*

$\forall \pi \in \mathit{Paths}(TS) : \forall m \geq 0 : \exists \pi' \in \mathit{Paths}(\pi[m]) : \exists n \geq 0 : \pi'[n] \models a.$

How to Modify the Logic?

$$\forall \pi \in \mathit{Paths}(\mathit{TS}) : \forall m \geq 0 : \exists \pi' \in \mathit{Paths}(\pi[m]) : \underbrace{\exists n \geq 0 : \pi'[n] \models a}_{\Diamond a}$$

How to Modify the Logic?

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Recall that $\pi \models \Diamond a$ expresses that path π satisfies formula $\Diamond a$.

Question

$? \models \exists \Diamond a$.

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$? \models \exists \Diamond a$.

Answer

There exists a path π starting in state s such that $\pi \models \Diamond a$, hence, $s \models \exists \Diamond a$.

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There exists a path π starting in state s such that $\pi \models \Diamond a$, hence, $s \models \exists \Diamond a$.

Consequence

We should distinguish between *path formulas* and *state formulas*.

Syntax

The *state formulas* are defined by

$$\Phi ::= \text{true} \mid a \mid \Phi \land \Phi \mid \neg\Phi \mid \exists\varphi \mid \forall\varphi$$

The *path formulas* are defined by

$$\varphi ::= \bigcirc\Phi \mid \Phi \, \mathsf{U} \, \Phi$$

Computation tree logic (CTL)

Edmund M. Clarke and E. Allen Emerson. Design and synthesis of synchronization skeletons using branching time temporal logic. In, Dexter Kozen, editor, *Proceedings of Workshop on Logic of Programs*, volume 131 of *Lecture Notes in Computer Science*, pages 52–71. Yorktown Heights, NY, USA, May 1981. Springer-Verlag.

Jean-Pierre Queille and Joseph Sifakis. Specification and verification of concurrent systems in CESAR. In, Mariangiola Dezani-Ciancaglini and Ugo Montanari, editors, *Proceedings of the 5th International Symposium on Programming*, volume 137 of *Lecture Notes in Computer Science*, pages 337–351. Torino, Italy, April 1982. Springer-Verlag.

Syntactic sugar

$$
\begin{aligned}
\exists\Diamond\Phi &= \exists(\text{true}\ \mathsf{U}\ \Phi)\\
\forall\Diamond\Phi &= \forall(\text{true}\ \mathsf{U}\ \Phi)\\
\exists\Box\Phi &= \neg\forall(\text{true}\ \mathsf{U}\ \neg\Phi)\\
\forall\Box\Phi &= \neg\exists(\text{true}\ \mathsf{U}\ \neg\Phi)
\end{aligned}
$$

Example

Question

How to express
“Each red light is preceded by a green light”
in CTL?

Example

Question

How to express
"Each red light is preceded by a green light"
in CTL?

Answer

$\neg \text{red} \wedge \forall \Box (\text{green} \vee \forall \bigcirc \neg \text{red})$

Example

Question

How to express
"The light is infinitely often green"
in CTL?

Example

Question

How to express
"The light is infinitely often green"
in CTL?

Answer

$\forall\Box\forall\Diamond$green

Semantics of CTL

$$
\begin{array}{rcl}
s \models \text{true} & & \\
s \models a & \text{iff} & a \in \ell(s) \\
s \models \Phi \wedge \Psi & \text{iff} & s \models \Phi \text{ and } s \models \Psi \\
s \models \neg\Phi & \text{iff} & \text{not}(s \models \Phi) \\
s \models \exists\varphi & \text{iff} & \exists\pi \in \mathit{Paths}(s) : \pi \models \varphi \\
s \models \forall\varphi & \text{iff} & \forall\pi \in \mathit{Paths}(s) : \pi \models \varphi
\end{array}
$$

and

$$
\begin{array}{rcl}
\pi \models \bigcirc\Phi & \text{iff} & \pi[1] \models \Phi \\
\pi \models \Phi \,\mathsf{U}\, \Psi & \text{iff} & \exists i \geq 0 : \pi[i] \models \Psi \text{ and } \forall 0 \leq j < i : \pi[j] \models \Phi
\end{array}
$$

Semantics of CTL

$$TS \models \Phi \text{ iff } \forall s \in I : s \models \Phi.$$

The *satisfaction set* $Sat(\Phi)$ is defined by

$$Sat(\Phi) = \{ s \in S \mid s \models \Phi \}.$$

Question

Recall that

$$\exists\Diamond\Phi = \exists(\text{true} \mathbin{\mathsf{U}} \Phi).$$

How is

$$s \models \exists\Diamond\Phi$$

defined?

Semantics of CTL

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Recall that

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Answer

$$\exists \pi \in Paths(s) : \exists i \geq 0 : \pi[i] \models \Phi.$$

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Answer

$$\forall \pi \in \mathit{Paths}(s) : \exists i \geq 0 : \pi[i] \models \Phi.$$

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Recall that

$$\exists \Box \Phi = \neg \forall (\text{true}\ \mathsf{U}\ \neg \Phi)$$

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Recall that

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$$\exists\pi \in \mathit{Paths}(s) : \forall i \geq 0 : \pi[i] \models \Phi.$$

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Answer

$$\forall \pi \in \mathit{Paths}(s) : \forall i \geq 0 : \pi[i] \models \Phi.$$

Theorem

The property

$$\forall \pi \in \mathit{Paths}(\mathit{TS}) : \forall m \geq 0 : \exists \pi' \in \mathit{Paths}(\pi[m]) : \exists n \geq 0 : \pi'[n] \models a$$

cannot be captured by LTL, but is captured by the CTL formula $\forall \Box \exists \Diamond a$.

Expressiveness of LTL and CTL

Theorem

The property

$$\forall \pi \in Paths(TS) : \exists i \geq 0 : \forall j \geq i : \pi[j..] \models a$$

cannot be captured by CTL, but is captured by the LTL formula $\Diamond \Box a$.