Binary Decision Diagrams

EECS 4315

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Model checking

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  Drawback: the state space of interesting systems is usually too large to represent explicitly.

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  Key idea: exploit the fact that the state space of most systems is not random.

  We focus on one symbolic approach:

  - BDD based

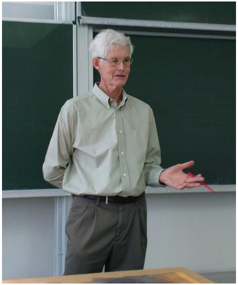
Satisfiability

Cook's theorem

Satisfiability checking of Boolean expressions is NP-complete.

Stephen Cook

- recipient of the ACM Turing award (1982)
- fellow of the Royal Society of London (1998)
- fellow of the Royal Society of Canada (1984)
- member of the National Academy of Sciences (1985)
- member of the American Academy of Arts and Sciences (1986)



Source: Jiri Janicek

Tautology

Theorem

Tautology checking of Boolean expressions is co-NP-complete.

Disjunctive normal form

Definition

A *literal* is a variable or its negation.

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If-then-else normal form

Notation

$$
\begin{array}{rcl}
0 & : & \text{false} \\
1 & : & \text{true} \\
x \to t_1, t_0 & : & (x \wedge t_1) \vee (\neg x \wedge t_0)
\end{array}
$$

Definition

The set of Boolean expressions in *if-then-else normal form (INF)* is defined by

$$t ::= 0 \mid 1 \mid x \to t, t$$

If-then-else normal form

Question

Give a Boolean expression in INF equivalent to $x_1 \wedge (\neg x_2 \vee x_3)$.

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Answer

$$
\begin{aligned}
t &= x_1 \to t_1, t_0 \\
t_0 &= x_2 \to t_{01}, t_{00} \\
t_1 &= x_2 \to t_{11}, t_{10} \\
t_{00} &= x_3 \to 0, 0 \\
t_{01} &= x_3 \to 0, 0 \\
t_{10} &= x_3 \to 1, 1 \\
t_{11} &= x_3 \to 1, 0
\end{aligned}
$$

If-then-else normal form

Shannon's expansion theorem

For every Boolean expression t and variable x,

$$t = x \to t[1/x], t[0/x].$$

Proposition

Any Boolean expression is equivalent to one in INF.

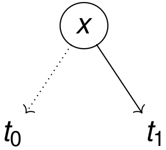
Decision trees

Boolean expressions in INF can be viewed as binary trees known as *decision trees*.

Two types of leaves: 0 and 1

$\boxed{0}$ $\boxed{1}$

One type of internal nodes: $x \rightarrow t_1, t_0$



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Draw the decision tree for the Boolean expression in INF equivalent to $x_1 \wedge (\neg x_2 \vee x_3)$.

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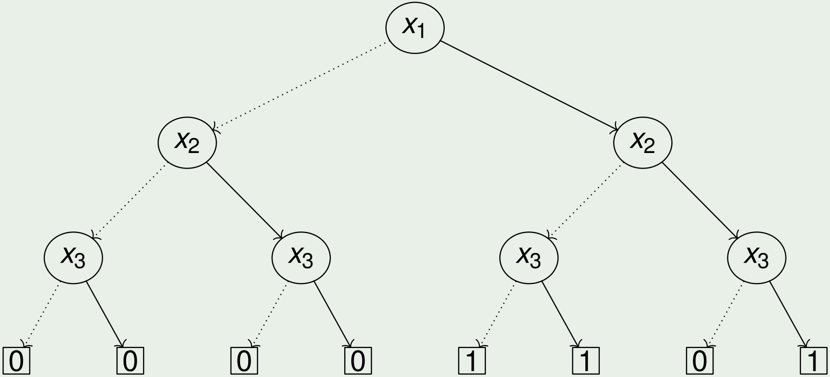

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Question

Identify all equal subexpressions.

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Question

Identify all equal subexpressions.

Answer

There are multiple occurrences of 0 and 1. Furthermore, t_{00} and t_{01} are equal.

Binary decision diagram

Question

Identify the equal subtrees in the decision tree for the Boolean expression in INF equivalent to $x_1 \wedge (\neg x_2 \vee x_3)$.

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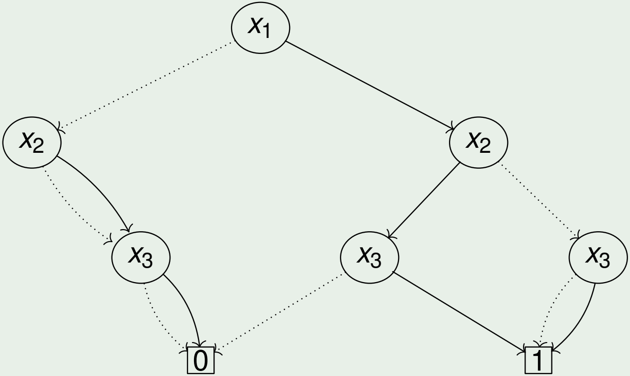


Binary decision diagram

Definition

A *binary decision diagram (BDD)* is a rooted directed acyclic graph where

- two (external) nodes where have out-degree zero and are labelled 0 and 1,
- and all other (internal) nodes have out-degree two, with one outgoing edge called the low edge and the other called the high edge, and are labelled with a variable.

Notation

Let u be an internal node.
$var(u)$ denotes the variable with which node u is labelled.
$low(u)$ denotes the successor of node u along its low edge (corresponding to the case that value of $var(u)$ is low, that is, 0).
$high(u)$ denotes the successor of node u along its high edge (corresponding to the case that value of $var(u)$ is high, that is, 1).

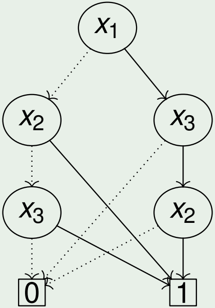
Ordered binary decision diagrams

Definition

A BDD is *ordered* if on all paths through the graph the variables respect a given linear order $x_1 < x_2 < \cdots < x_n$.

Question

Is the BDD



ordered?

Reduced ordered binary decision diagrams

Definition

An ordered BDD is *reduced* if

- *unique*: no two distinct internal nodes u and v have the same variable, low- and high-successor, that is,

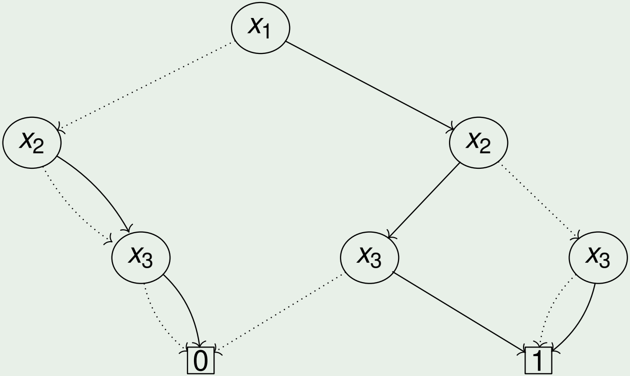
  if $\mathit{var}(v) = \mathit{var}(u), \mathit{low}(v) = \mathit{low}(u),$ and $\mathit{high}(v) = \mathit{high}(u)$ then $u = v$.

- *non-redundant*: no internal node u has identical low- and high-successor, that is,

$$\mathit{low}(u) \neq \mathit{high}(u).$$

Question

Is the ordered BDD



reduced?

Question

What is the corresponding reduced ordered BDD?

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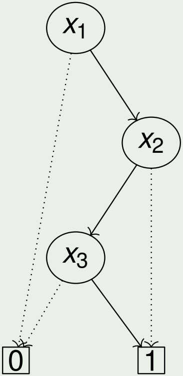

Canonicity lemma

Lemma

For a Boolean expression t with variables $x_1, x_2, \ldots, x_n$ and a linear order $x_1 < x_2 < \cdots < x_n$, there exists a unique reduced ordered BDD which is equivalent to t.

For the remainder, we restrict our attention to reduced ordered BDDs and simply call them BDDs.

- member of the National Academy of Engineering (2003),
- recipient of the Paris Kanellakis Theory and Practice Award (1997)
- recipient of the IEEE Emanuel R. Piore Award (2007)
- his paper on BDDs is one of the most cited computer science papers (more than 9660 citations)

Source: Randal Bryant

BDDs

Proposition

Satisfiability checking of BDDs is constant time.

Proposition

Tautology checking of BDDs is constant time.

The variable order matters

Question

Draw the BDD corresponding to

$$(x_1 \wedge x_2) \vee (x_3 \wedge x_4) \vee (x_5 \wedge x_6)$$

for the variable ordering

$$x_1 < x_2 < x_3 < x_4 < x_5 < x_6$$

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The variable order matters

Theorem

Deciding whether a given variable order is optimal is NP-hard.

Heuristics are used to find good variable orderings. For more details, see, for example,
I. Wegener. *Branching Programs and Binary Decision Diagrams: Theory and Applications.* 2000.