Linear Temporal Logic

EECS 4315

www.eecs.yorku.ca/course/4315/

Linear Temporal Logic

Linear temporal logic (LTL) is a logic to reason about systems with nondeterminism.

The logic was introduced by Amir Pnueli.

A. Pnueli. The temporal logic of programs. In *Proceedings of the 18th IEEE Symposium on Foundations of Computer Science*, pages 46–67. Providence, RI, USA, October/November 1977. IEEE.

Amir Pnueli (1941–2009)

- Recipient of the Turing Award (1996)
- Recipient of the Israel prize (2000)
- Foreign Associate of the U.S. National Academy of Engineering (1999)
- Fellow of the Association for Computing Machinery (2007)

Source: David Monniaux

Linear Temporal Logic

Definition

LTL is defined by the following grammar.

$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \, \mathsf{U} \, f$$

where a is an atomic proposition.

An atomic proposition represents a basic property (such as the value of a particular variable being even).

LTL

Given an execution path p, does it satisfy a particular LTL formula f?

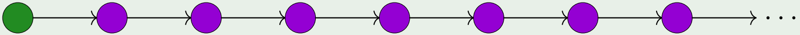
An atomic proposition a is satisfied if a holds in the initial state of the execution path.

Given an execution path p, does it satisfy a particular LTL formula f?

An atomic proposition a is satisfied if a holds in the initial state of the execution path.

Question

Does the execution path



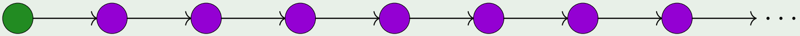
satisfy the atomic proposition green?

Given an execution path p, does it satisfy a particular LTL formula f?

An atomic proposition a is satisfied if a holds in the initial state of the execution path.

Question

Does the execution path



satisfy the atomic proposition green?

Answer

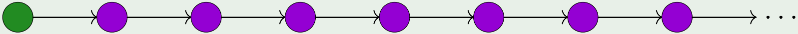
Yes.

Given an execution path p, does it satisfy a particular LTL formula f?

An atomic proposition a is satisfied if a holds in the initial state of the execution path.

Question

Does the execution path



satisfy the atomic proposition purple?

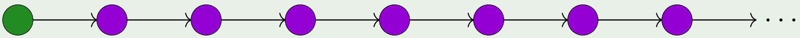
LTL

Given an execution path p, does it satisfy a particular LTL formula f?

An atomic proposition a is satisfied if a holds in the initial state of the execution path.

Question

Does the execution path



satisfy the atomic proposition purple?

Answer

No.

LTL

Given an execution path p, does it satisfy a particular LTL formula f?

The LTL formula $\bigcirc a$ (pronounced as next a) is satisfied if a holds in the next state of the execution path (that is, the second state).

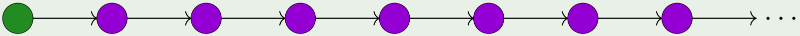
LTL

Given an execution path p, does it satisfy a particular LTL formula f?

The LTL formula $\bigcirc a$ (pronounced as next a) is satisfied if a holds in the next state of the execution path (that is, the second state).

Question

Does the execution path



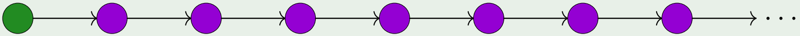
satisfy the atomic proposition $\bigcirc$ green?

Given an execution path p, does it satisfy a particular LTL formula f?

The LTL formula $\bigcirc a$ (pronounced as next a) is satisfied if a holds in the next state of the execution path (that is, the second state).

Question

Does the execution path



satisfy the atomic proposition $\bigcirc$ green?

Answer

No.

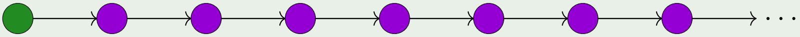
LTL

Given an execution path p, does it satisfy a particular LTL formula f?

The LTL formula $\bigcirc a$ (pronounced as next a) is satisfied if a holds in the next state of the execution path (that is, the second state).

Question

Does the execution path



satisfy the atomic proposition $\bigcirc$ purple?

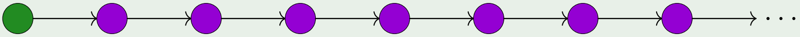
LTL

Given an execution path p, does it satisfy a particular LTL formula f?

The LTL formula $\bigcirc a$ (pronounced as next a) is satisfied if a holds in the next state of the execution path (that is, the second state).

Question

Does the execution path



satisfy the atomic proposition $\bigcirc$ purple?

Answer

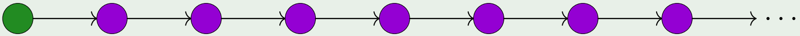
Yes.

Given an execution path *p*, does it satisfy a particular LTL formula *f*?

The LTL formula $\bigcirc a$ (pronounced as next *a*) is satisfied if *a* holds in the next state of the execution path (that is, the second state).

Question

Does the execution path



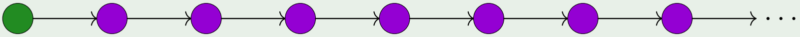
satisfy the atomic proposition $\bigcirc\bigcirc$ purple?

Given an execution path p, does it satisfy a particular LTL formula f?

The LTL formula $\bigcirc a$ (pronounced as next a) is satisfied if a holds in the next state of the execution path (that is, the second state).

Question

Does the execution path



satisfy the atomic proposition $\bigcirc\bigcirc$ purple?

Answer

Yes.

LTL

Given an execution path p, does it satisfy a particular LTL formula f?

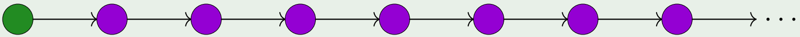
The LTL formula $a \, \mathrm{U} \, b$ (pronounced as a until b) is satisfied if b holds in some state of the execution path and a holds in all states before that state.

Given an execution path p, does it satisfy a particular LTL formula f?

The LTL formula $a \, \mathrm{U} \, b$ (pronounced as a until b) is satisfied if b holds in some state of the execution path and a holds in all states before that state.

Question

Does the execution path



satisfy the atomic proposition green U purple?

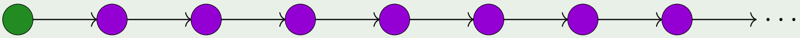
LTL

Given an execution path *p*, does it satisfy a particular LTL formula *f*?

The LTL formula *a* U *b* (pronounced as *a* until *b*) is satisfied if *b* holds in some state of the execution path and *a* holds in all states before that state.

Question

Does the execution path



satisfy the atomic proposition green U purple?

Answer

Yes.

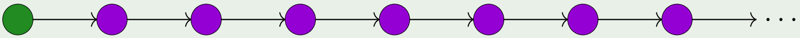
LTL

Given an execution path p, does it satisfy a particular LTL formula f?

The LTL formula $a \, \mathrm{U} \, b$ (pronounced as a until b) is satisfied if b holds in some state of the execution path and a holds in all states before that state.

Question

Does the execution path



satisfy the atomic proposition purple U green?

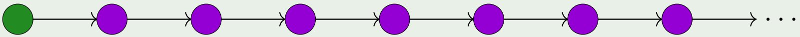
LTL

Given an execution path *p*, does it satisfy a particular LTL formula *f*?

The LTL formula *a* U *b* (pronounced as *a* until *b*) is satisfied if *b* holds in some state of the execution path and *a* holds in all states before that state.

Question

Does the execution path



satisfy the atomic proposition purple U green?

Answer

Yes.

Syntactic sugar

As usual

$$\begin{array}{rcl} \text{true} & = & a \vee \neg a \\ f_1 \vee f_2 & = & \neg(\neg f_1 \wedge \neg f_2) \\ f_1 \Rightarrow f_2 & = & \neg f_1 \vee f_2 \end{array}$$

Also

$$\begin{array}{rcll} \Diamond f & = & \text{true}\ \mathrm{U}\ f & \text{(eventually } f\text{)} \\ \Box f & = & \neg \Diamond \neg f & \text{(always } f\text{)} \end{array}$$

Alternative syntax

$$\begin{array}{lcl} \mathsf{X}f & : & \bigcirc f \\ \mathsf{F}f & : & \Diamond f \\ \mathsf{G}f & : & \Box f \end{array}$$

State Space Diagram

Question

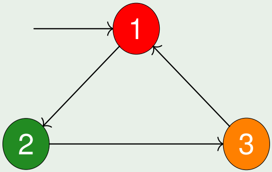
Draw the state space diagram of a model of a traffic light.

State Space Diagram

Question

Draw the state space diagram of a model of a traffic light.

Answer



Note: the transitions are not labelled, but the states are labelled.

Transition System

Definition

A transition system is a tuple $\langle S, L, I, \rightarrow, \ell \rangle$ consisting of

- a set S of states,
- a set L of labels,
- A set $I \subseteq S$ of initial states,
- a transition relation $\rightarrow \subseteq S \times S$ such that for all $s \in S$ there exists a $s' \in S$ such that $s \rightarrow s'$, and
- a labelling function $\ell : S \rightarrow 2^L$.

2^L denotes the set of subsets of L.

Powerset

Question

What is $2^{\{1,2,3\}}$?

Powerset

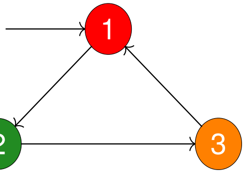
Question

What is $2^{\{1,2,3\}}$?

Answer

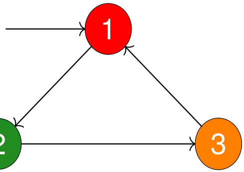
$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Transition System



Question

Give the transition system modelling a traffic light.

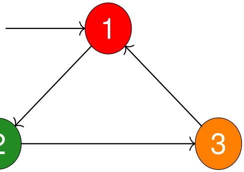


Question

Give the transition system modelling a traffic light.

Answer

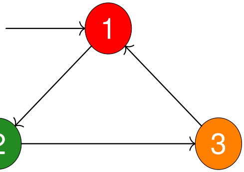
$$\langle\{1,2,3\},\{\text{red},\text{green},\text{orange}\},$$
$$\{1\},\{1\rightarrow 2, 2\rightarrow 3, 3\rightarrow 1\}$$
$$\{1\mapsto\{\text{red}\}, 2\mapsto\{\text{green}\}, 3\mapsto\{\text{orange}\}\}\rangle$$



$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \,\mathsf{U}\, f$$

Question

Which LTL formula expresses
“initially the light is red.”



$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \, \mathsf{U} \, f$$

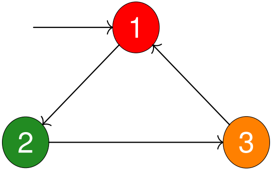
Question

Which LTL formula expresses
"initially the light is red."

Answer

The LTL formula red.

LTL

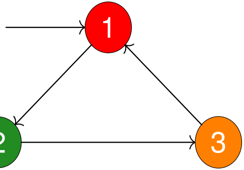


$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \, \mathsf{U} \, f$$

Question

Which LTL formula expresses

"initially the light is red and next it becomes green."



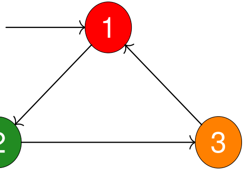
$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \, \mathsf{U} \, f$$

Question

Which LTL formula expresses
"initially the light is red and next it becomes green."

Answer

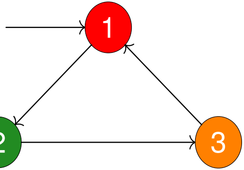
The LTL formula red $\wedge$ $\bigcirc$green



$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \, \mathsf{U} \, f$$

Question

Which LTL formula expresses
“the light becomes eventually orange.”



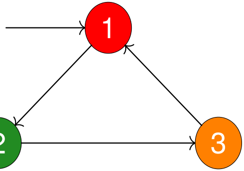
$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \ \mathrm{U}\ f$$

Question

Which LTL formula expresses
“the light becomes eventually orange.”

Answer

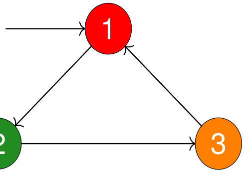
The LTL formula true U orange $=$ $\Diamond$orange



$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \,\mathrm{U}\, f$$

Question

Which LTL formula expresses
"the light is infinitely often red."



$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \, \mathsf{U} \, f$$

Question

Which LTL formula expresses
"the light is infinitely often red."

Answer

The LTL formula $\Box\Diamond$red

Question

What does the LTL formula $\Box(\text{green} \Rightarrow \neg \bigcirc \text{red})$ express?

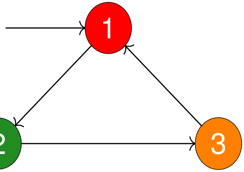
Question

What does the LTL formula $\Box(\text{green} \Rightarrow \neg \bigcirc \text{red})$ express?

Answer

"Once green, the light cannot become red immediately"

Execution Paths



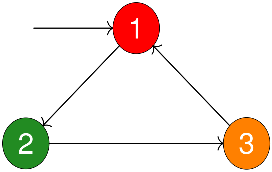
Definition

Paths(s) is the set of path starting in state s.

Question

What is *Paths*(2)?

Execution Paths



Definition

Paths(s) is the set of path starting in state s.

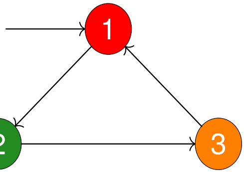
Question

What is *Paths*(2)?

Answer

$\{231231231231\cdots\}$

Execution Paths



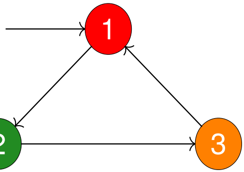
Definition

Let $p \in Paths(s)$ and $n \geq 0$. Then $p[n]$ is the $(n+1)^{\text{th}}$ state of the path p.

Question

Let $p = 123123\cdots$. What is $p[3]$?

Execution Paths



Definition

Let $p \in \mathit{Paths}(s)$ and $n \geq 0$. Then $p[n]$ is the $(n+1)^{\text{th}}$ state of the path p.

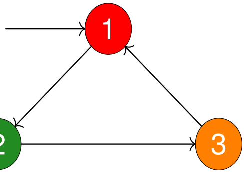
Question

Let $p = 123123\cdots$. What is $p[3]$?

Answer

1.

Execution Paths



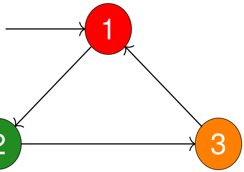
Definition

Let $p \in \mathit{Paths}(s)$ and $n \geq 0$. Then $p[n..]$ is the suffix of p starting with the $(n+1)^{\text{th}}$ state.

Question

Let $p = 123123\cdots$. What is $p[2..]$?

Execution Paths



Definition

Let $p \in \mathit{Paths}(s)$ and $n \geq 0$. Then $p[n..]$ is the suffix of p starting with the $(n+1)^{\text{th}}$ state.

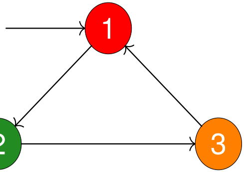
Question

Let $p = 123123\cdots$. What is $p[2..]$?

Answer

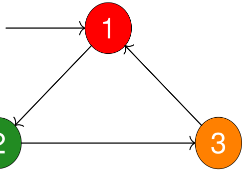
$312312\cdots$

Semantics of LTL



$p \models f$ denotes that path p satisfies LTL formula f

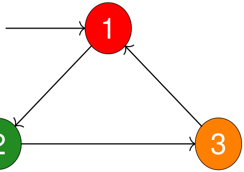
Semantics of LTL



$p \models f$ denotes that path p satisfies LTL formula f

Question

$123123 \cdots \models$ green?



$p \models f$ denotes that path p satisfies LTL formula f

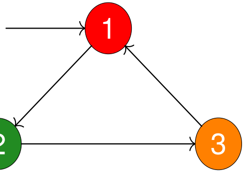
Question

$123123\cdots \models$ green?

Answer

No.

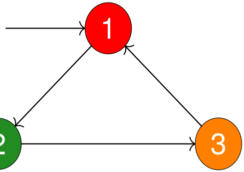
Semantics of LTL



$p \models f$ denotes that path p satisfies LTL formula f

Question

$123123 \cdots \models \bigcirc$green?



$p \models f$ denotes that path p satisfies LTL formula f

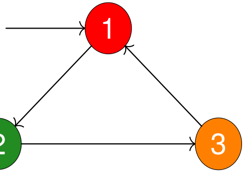
Question

$123123 \cdots \models \bigcirc$green?

Answer

Yes.

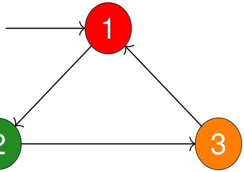
Semantics of LTL



$p \models f$ denotes that path p satisfies LTL formula f

Question

$123123 \cdots \models$ red $\wedge \bigcirc$green?



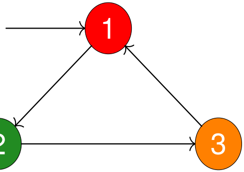
$p \models f$ denotes that path p satisfies LTL formula f

Question

$123123\cdots \models$ red $\wedge \bigcirc$green?

Answer

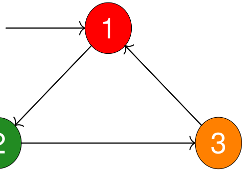
Yes.



$p \models f$ denotes that path p satisfies LTL formula f

Question

$123123 \cdots \models \neg$green?



$p \models f$ denotes that path p satisfies LTL formula f

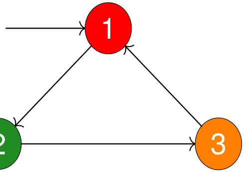
Question

$123123\cdots \models \neg$green?

Answer

Yes.

Semantics of LTL

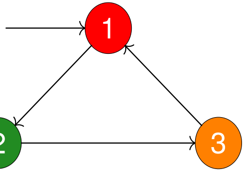


$p \models f$ denotes that path p satisfies LTL formula f

Question

$123123 \cdots \models$ red U green?

Semantics of LTL



$p \models f$ denotes that path p satisfies LTL formula f

Question

$123123\cdots \models$ red U green?

Answer

Yes.

Semantics of LTL

Definition

$$
\begin{array}{rcl}
p \models a & \text{iff} & a \in \ell(p[0]) \\
p \models f \wedge \psi & \text{iff} & p \models f \wedge p \models \psi \\
p \models \neg f & \text{iff} & p \not\models f \\
p \models \bigcirc f & \text{iff} & p[1..] \models f \\
p \models f \,\mathsf{U}\, \psi & \text{iff} & \exists i \geq 0 : p[i..] \models \psi \wedge \forall 0 \leq j < i : p[j..] \models f
\end{array}
$$

Semantics of LTL

$$TS \models f \text{ iff } \forall s \in I : s \models f$$

where

$$s \models f \text{ iff } \forall p \in Paths(s) : p \models f$$

where

$$\begin{array}{rcl} p \models a & \text{iff} & a \in L(p[0]) \\ p \models f \wedge \psi & \text{iff} & p \models f \wedge p \models \psi \\ p \models \neg f & \text{iff} & p \not\models f \\ p \models \bigcirc f & \text{iff} & p[1..] \models f \\ p \models f \,\mathsf{U}\, \psi & \text{iff} & \exists i \geq 0 : p[i..] \models \psi \wedge \forall 0 \leq j < i : p[j..] \models f \end{array}$$