Linear Temporal Logic

EECS 4315

www.eecs.yorku.ca/course/4315/

Definition

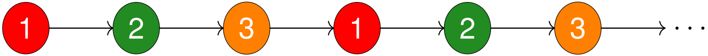
LTL is defined by the following grammar.

$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \, \mathrm{U} \, f$$

where a is an atomic proposition.

Semantics of LTL

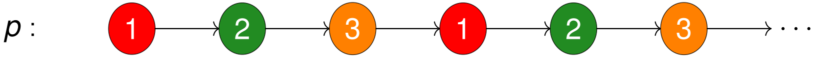
p :



Question

$p \models$ red?

Semantics of LTL



Question

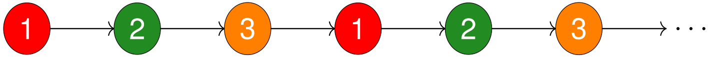
$p \models$ red?

Answer

Yes.

Semantics of LTL

p :

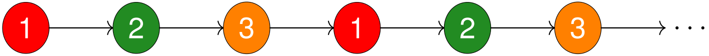


Question

$p \models \bigcirc\bigcirc$red?

Semantics of LTL

p :



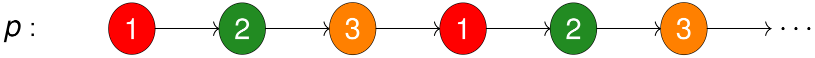
Question

$p \models \bigcirc\bigcirc \text{red}$?

Answer

No.

Semantics of LTL



Question

$p \models$ red U green?

Semantics of LTL

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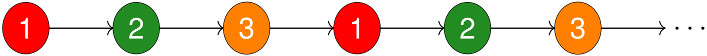

Question

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Answer

Yes.

Semantics of LTL

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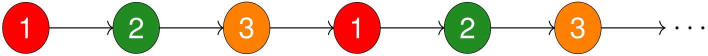
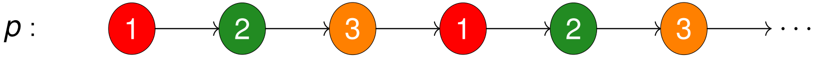


Question

$p \models \Diamond$green?

Semantics of LTL



Question

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Answer

Yes.

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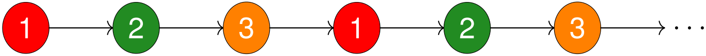

Question

$p \models \Box\neg$red?

Semantics of LTL

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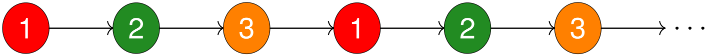

Question

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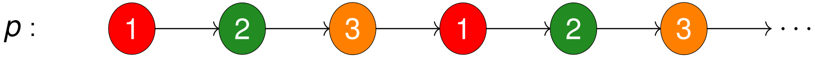
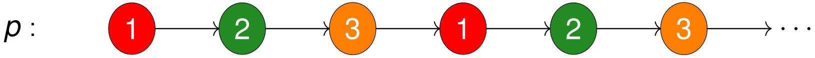
No.

Semantics of LTL



Question

$p \models (\Diamond \text{green})\ \mathsf{U}\ (\bigcirc \text{red})$?



Question

$p \models (\Diamond \text{green})\ \mathsf{U}\ (\bigcirc \text{red})$?

Answer

Yes.

Semantics of LTL

$$TS \models f \text{ iff } \forall s \in I : s \models f$$

where

$$s \models f \text{ iff } \forall p \in Paths(s) : p \models f$$

where

$$\begin{array}{rcl} p \models a & \text{iff} & a \in L(p[0]) \\ p \models f_1 \wedge f_2 & \text{iff} & p \models f_1 \wedge p \models f_2 \\ p \models \neg f & \text{iff} & p \not\models f \\ p \models \bigcirc f & \text{iff} & p[1..] \models f \\ p \models f_1 \text{ U } f_2 & \text{iff} & \exists i \geq 0 : p[i..] \models f_2 \wedge \forall 0 \leq j < i : p[j..] \models f_1 \end{array}$$

Semantics of LTL

Question

When $p \models \Diamond f$?

Semantics of LTL

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Answer

$\exists j \geq 0 : p[j..] \models f$.

Question

When $p \models \Box f$?

Semantics of LTL

Question

When $p \models \Box f$?

Answer

$\forall j \geq 0 : p[j..] \models f$.

Equivalence

Definition

The LTL formulas f and g are equivalent, denoted $f \equiv g$, if for all transition systems TS,

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Exercise

Are the following formulas equivalent? Either provide a proof or a counter example.

(a) $\Diamond(f \wedge g) \equiv \Diamond f \wedge \Diamond g$?

(b) $\Diamond \bigcirc f \equiv \bigcirc \Diamond f$?

Invariants

Definition

The class of LTL formulas that capture *invariants* is defined by $\Box g$, where

$$g ::= a \mid g \wedge g \mid \neg g.$$

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Example

$\Box\neg$red.

Safety properties

Safety properties are characterized by “nothing bad ever happens.” For example, “a red light is immediately preceded by orange” is a safety property.

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Question

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Answer

$\Box(\bigcirc \text{red} \Rightarrow \text{orange})$.

Liveness properties

Liveness properties are characterized by “something good eventually happens.” For example, “the light is infinitely often red” is a liveness property.

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Answer

$\Box\Diamond$red.

Question

Are there properties we cannot express in LTL?

Expressiveness of LTL

Question

Are there properties we cannot express in LTL?

Answer

Yes, for example, “Always a state satisfying *a* can be reached”

Expressiveness of LTL

Theorem

There does not exists an LTL formula φ with $TS \models \varphi$ iff

$$\forall p \in Paths(TS) : \forall m \geq 0 : \exists p' \in Paths(p[m]) : \exists n \geq 0 : p'[n] \models a.$$

How to Modify the Logic?

$$\forall p \in Paths(TS) : \forall m \geq 0 : \exists p' \in Paths(p[m]) : \underbrace{\exists n \geq 0 : p'[n] \models a}_{\Diamond a}$$

How to Modify the Logic?

$$\forall p \in \mathit{Paths}(TS) : \forall m \geq 0 : \overbrace{\exists p' \in \mathit{Paths}(p[m]) : \underbrace{\exists n \geq 0 : p'[n] \models a}_{\Diamond a}}^{\exists \Diamond a}$$

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Recall that $p \models \Diamond a$ expresses that path p satisfies formula $\Diamond a$.

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Answer

There exists a path p starting in state s such that $p \models \Diamond a$, hence, $s \models \exists \Diamond a$.

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There exists a path p starting in state s such that $p \models \Diamond a$, hence, $s \models \exists \Diamond a$.

Consequence

We should distinguish between *path formulas* and *state formulas*.

Syntax

The *state formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists g \mid \forall g$$

The *path formulas* are defined by

$$g ::= \bigcirc f \mid f \, \mathsf{U} \, f$$

Computation tree logic (CTL)

Edmund M. Clarke and E. Allen Emerson. Design and synthesis of synchronization skeletons using branching time temporal logic. In, Dexter Kozen, editor, *Proceedings of Workshop on Logic of Programs*, volume 131 of *Lecture Notes in Computer Science*, pages 52–71. Yorktown Heights, NY, USA, May 1981. Springer-Verlag.

Jean-Pierre Queille and Joseph Sifakis. Specification and verification of concurrent systems in CESAR. In, Mariangiola Dezani-Ciancaglini and Ugo Montanari, editors, *Proceedings of the 5th International Symposium on Programming*, volume 137 of *Lecture Notes in Computer Science*, pages 337–351. Torino, Italy, April 1982. Springer-Verlag.

Syntax

The *state formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists \bigcirc f \mid \exists (f \, \mathsf{U} \, f) \mid \forall \bigcirc f \mid \forall (f \, \mathsf{U} \, f)$$