Logics EECS 4315

www.eecs.yorku.ca/course/4315/

Semantics of LTL

$$TS \models f \text{ iff } \forall s \in I : s \models f$$

where

$$s \models f \text{ iff } \forall p \in Paths(s) : p \models f$$

where

$$\begin{array}{cccc} p \models a & \mathrm{iff} & a \in L(p[0]) \\ p \models f_1 \wedge f_2 & \mathrm{iff} & p \models f_1 \wedge p \models f_2 \\ p \models \neg f & \mathrm{iff} & p \not\models f \\ p \models \bigcirc f & \mathrm{iff} & p[1..] \models f \\ p \models f_1 \cup f_2 & \mathrm{iff} & \exists i \geq 0 : p[i..] \models f_2 \wedge \forall 0 \leq j < i : p[j..] \models f_1 \end{array}$$

Equivalence

Definition

The LTL formulas f and g are equivalent, denoted $f \equiv g$, if for all transition systems TS,

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Exercise

Are the following formulas equivalent? Either provide a proof or a counter example.

(a)
$$\Diamond (f \wedge g) \equiv \Diamond f \wedge \Diamond g$$
?

(b)
$$\Diamond \bigcirc f \equiv \bigcirc \Diamond f$$
?

Invariants

Definition

The class of LTL formulas that capture *invariants* is defined by $\Box g$, where

$$g ::= a \mid g \wedge g \mid \neg g$$
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Example

 $\Box \neg red.$

Safety properties

Safety properties are characterized by "nothing bad ever happens." For example, "a red light is immediately preceded by orange" is a safety property.

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Question

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Answer

 \Box (\bigcirc red \Rightarrow orange).

Liveness properties

Liveness properties are characterized by "something good eventually happens." For example, "the light is infinitely often red" is a liveness property.

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Answer

□◊red.

Expressiveness of LTL

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Are there properties we cannot express in LTL?

Expressiveness of LTL

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Are there properties we cannot express in LTL?

Answer

Yes, for example, "Always a state satisfying a can be reached"

Expressiveness of LTL

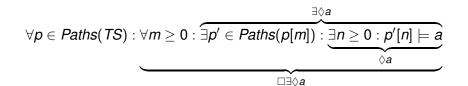
Theorem

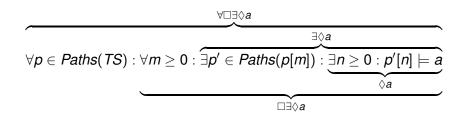
There does not exists an LTL formula φ with $TS \models \varphi$ iff

 $\forall p \in Paths(TS) : \forall m \geq 0 : \exists p' \in Paths(p[m]) : \exists n \geq 0 : p'[n] \models a.$

$$\forall p \in Paths(TS) : \forall m \geq 0 : \exists p' \in Paths(p[m]) : \underbrace{\exists n \geq 0 : p'[n] \models a}_{\land a}$$

$$\forall p \in Paths(TS) : \forall m \geq 0 : \overbrace{\exists p' \in Paths(p[m]) : \underbrace{\exists n \geq 0 : p'[n] \models a}_{\Diamond a}}$$





$$\overbrace{\exists p' \in \textit{Paths}(p[m]) : \underbrace{\exists n \geq 0 : p'[n] \models a}_{\Diamond a}}^{\exists \Diamond a}$$

Recall that $p \models \Diamond a$ expresses that path p satisfies formula $\Diamond a$.

Question

? \models ∃ \Diamond a.

$$\overbrace{\exists p' \in \textit{Paths}(p[m]) : \underbrace{\exists n \geq 0 : p'[n] \models a}_{\Diamond a}}^{\exists \Diamond a}$$

Recall that $p \models \Diamond a$ expresses that path p satisfies formula $\Diamond a$.

Question

? |= ∃◊*a*.

Answer

There exists a path p starting in state s such that $p \models \Diamond a$, hence, $s \models \exists \Diamond a$.

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Recall that $p \models \Diamond a$ expresses that path p satisfies formula $\Diamond a$.

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Answer

There exists a path p starting in state s such that $p \models \Diamond a$, hence, $s \models \exists \Diamond a$.

Consequence

We should distinguish between *path formulas* and *state formulas*.

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Syntax

The state formulas are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists g \mid \forall g$$

The path formulas are defined by

$$g ::= \bigcirc f \mid f \cup f$$

Computation Tree Logic

Computation tree logic (CTL)

Edmund M. Clarke and E. Allen Emerson. Design and synthesis of synchronization skeletons using branching time temporal logic. In, Dexter Kozen, editor, *Proceedings of Workshop on Logic of Programs*, volume 131 of *Lecture Notes in Computer Science*, pages 52–71. Yorktown Heights, NY, USA, May 1981. Springer-Verlag.

Jean-Pierre Queille and Joseph Sifakis. Specification and verification of concurrent systems in CESAR. In, Mariangiola Dezani-Ciancaglini and Ugo Montanari, editors, *Proceedings of the 5th International Symposium on Programming*, volume 137 of *Lecture Notes in Computer Science*, pages 337–351. Torino, Italy, April 1982. Springer-Verlag.

Syntax

The state formulas are defined by

$$f ::= a \mid f \land f \mid \neg f \mid \exists \bigcirc f \mid \exists (f \cup f) \mid \forall \bigcirc f \mid \forall (f \cup f)$$

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Syntactic sugar

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\exists \lozenge f = \exists (\text{true U } f)
\forall \lozenge f = \forall (\text{true U } f)
\exists \Box f = \neg \forall (\text{true U } \neg f)
\forall \Box f = \neg \exists (\text{true U } \neg f)
```

Example

Question

How to express "Each red light is preceded by a green light" in CTL?

Example

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How to express "Each red light is preceded by a green light" in CTL?

Answer

 $\neg \text{red} \land \forall \Box (\text{green} \lor \forall \bigcirc \neg \text{red})$