Computation Tree Logic

EECS 4315

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Computation Tree Logic

The *state formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists g \mid \forall g$$

The *path formulas* are defined by

$$g ::= \bigcirc f \mid f \,\mathrm{U}\, f$$

Syntactic sugar

$$\begin{aligned}
\exists\Diamond f &= \exists(\text{true U } f)\\
\forall\Diamond f &= \forall(\text{true U } f)\\
\exists\Box f &= \neg\forall(\text{true U } \neg f)\\
\forall\Box f &= \neg\exists(\text{true U } \neg f)
\end{aligned}$$

Example

Question

How to express “Each red light is preceded by a green light” in CTL?

Example

Question

How to express "Each red light is preceded by a green light" in CTL?

Answer

$\neg red \wedge \forall\Box(green \vee \forall\bigcirc\neg red)$

Example

Question

How to express “The light is infinitely often green” in CTL?

Example

Question

How to express “The light is infinitely often green” in CTL?

Answer

$\forall\Box\forall\Diamond$green

Semantics of CTL

$$\begin{array}{rcl} s \models a & \text{iff} & a \in \ell(s) \\ s \models f_1 \wedge f_2 & \text{iff} & s \models f_1 \text{ and } s \models f_2 \\ s \models \neg f & \text{iff} & \text{not}(s \models f) \\ s \models \exists g & \text{iff} & \exists p \in \mathit{Paths}(s) : p \models g \\ s \models \forall g & \text{iff} & \forall p \in \mathit{Paths}(s) : p \models g \end{array}$$

and

$$\begin{array}{rcl} p \models \bigcirc f & \text{iff} & p[1] \models f \\ p \models f_1 \,\mathsf{U}\, f_2 & \text{iff} & \exists i \geq 0 : p[i] \models f_2 \text{ and } \forall 0 \leq j < i : p[j] \models f_1 \end{array}$$

Semantics of CTL

$$TS \models f \text{ iff } \forall s \in I : s \models f.$$

The *satisfaction set* $Sat(f)$ is defined by

$$Sat(f) = \{ s \in S \mid s \models f \}.$$

Semantics of CTL

Question

Recall that

$$\exists \Diamond f = \exists(\text{true} \ \mathsf{U}\ f).$$

How is

$$s \models \exists \Diamond f$$

defined?

Semantics of CTL

Question

Recall that

$$\exists\Diamond f = \exists(\text{true U } f).$$

How is

$$s \models \exists\Diamond f$$

defined?

Answer

$$\exists p \in Paths(s) : \exists i \geq 0 : p[i] \models f.$$

Semantics of CTL

Question

Recall that

$$\forall\Diamond f = \forall(\text{true } \mathrm{U}\ f)$$

How is

$$s \models \forall\Diamond f$$

defined?

Semantics of CTL

Question

Recall that

$$\forall\Diamond f = \forall(\text{true U } f)$$

How is

$$s \models \forall\Diamond f$$

defined?

Answer

$$\forall p \in Paths(s) : \exists i \geq 0 : p[i] \models f.$$

Question

Recall that

$$\exists\Box f = \neg\forall(\text{true}\ \mathsf{U}\ \neg f)$$

How is

$$s \models \exists\Box f$$

defined?

Question

Recall that

$$\exists\Box f = \neg\forall(\text{true}\ \text{U}\ \neg f)$$

How is

$$s \models \exists\Box f$$

defined?

Answer

$$\exists p \in \mathit{Paths}(s) : \forall i \geq 0 : p[i] \models f.$$

Question

Recall that

$$\forall\Box f = \neg\exists(\text{true}\ \mathsf{U}\ \neg f)$$

How is

$$s \models \forall\Box f$$

defined?

Question

Recall that

$$\forall\Box f = \neg\exists(\text{true } \mathsf{U} \, \neg f)$$

How is

$$s \models \forall\Box f$$

defined?

Answer

$$\forall p \in Paths(s) : \forall i \geq 0 : p[i] \models f.$$

Expressiveness of LTL and CTL

Theorem

The property

$$\forall p \in Paths(TS) : \forall m \geq 0 : \exists p' \in Paths(p[m]) : \exists n \geq 0 : p'[n] \models a$$

cannot be captured by LTL, but is captured by the CTL formula $\forall \Box \exists \Diamond a$.

Theorem

The property

$$\forall p \in Paths(TS) : \exists i \geq 0 : \forall j \geq i : p[j..] \models a$$

cannot be captured by CTL, but is captured by the LTL formula $\Diamond\Box a$.

Model checking CTL

Basic idea

Compute $Sat(f)$ by recursion on the structure of f.

$TS \models f$ iff $I \subseteq Sat(f)$.

Alternative view

Label each state with the subformulas of f that it satisfies.

Definition

The *formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists \bigcirc f \mid \exists (f \, \mathsf{U} \, f) \mid \forall \bigcirc f \mid \forall (f \, \mathsf{U} \, f)$$

Question

What is $Sat(a)$?

Model checking CTL

Definition

The *formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists \bigcirc f \mid \exists (f \, \mathsf{U} \, f) \mid \forall \bigcirc f \mid \forall (f \, \mathsf{U} \, f)$$

Question

What is $Sat(a)$?

Answer

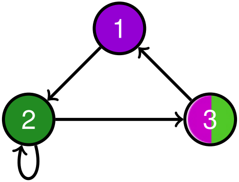
$Sat(a) = \{ s \in S \mid a \in \ell(s) \}$

Alternative view

Label each state s satisfying $a \in \ell(s)$ with a.

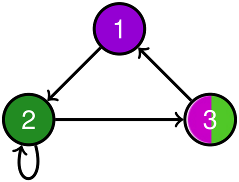
Example

green



Example

green



$$
\begin{aligned}
1 &\mapsto \emptyset \\
2 &\mapsto \{\text{green}\} \\
3 &\mapsto \{\text{green}\}
\end{aligned}
$$

Model checking CTL

Definition

The *formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists \bigcirc f \mid \exists (f \mathrm{U} f) \mid \forall \bigcirc f \mid \forall (f \mathrm{U} f)$$

Question

What is $Sat(f_1 \wedge f_2)$?

Model checking CTL

Definition

The *formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists \bigcirc f \mid \exists (f \: \mathsf{U} \: f) \mid \forall \bigcirc f \mid \forall (f \: \mathsf{U} \: f)$$

Question

What is $Sat(f_1 \wedge f_2)$?

Answer

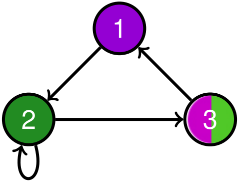
$Sat(f_1 \wedge f_2) = Sat(f_1) \cap Sat(f_2)$

Alternative view

Label states, that are labelled with both f_1 and f_2, also with $f_1 \wedge f_2$.

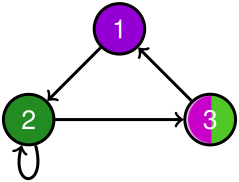
Example

green $\wedge$ purple



Example

green ∧ purple



$$\begin{array}{lcl} 1 & \mapsto & \{\text{purple}\} \\ 2 & \mapsto & \{\text{green}\} \\ 3 & \mapsto & \{\text{green}, \text{purple}, \text{green} \wedge \text{purple}\} \end{array}$$

Model checking CTL

Definition

The *formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists \bigcirc f \mid \exists (f \, \mathsf{U} \, f) \mid \forall \bigcirc f \mid \forall (f \, \mathsf{U} \, f)$$

Question

What is $Sat(\neg f)$?

Definition

The *formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists \bigcirc f \mid \exists (f \; \mathsf{U} \; f) \mid \forall \bigcirc f \mid \forall (f \; \mathsf{U} \; f)$$

Question

What is $Sat(\neg f)$?

Answer

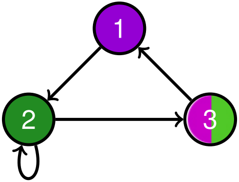
$Sat(\neg f) = S \setminus Sat(f)$

Alternative view

Label each state, that is not labelled with f, with $\neg f$.

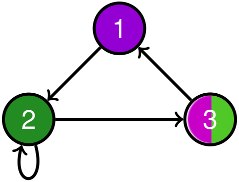
Example

$\neg(\text{green} \wedge \text{purple})$



Example

$$\neg(\text{green} \wedge \text{purple})$$



$$\begin{array}{lll} 1 & \mapsto & \{\text{purple}, \neg(\text{green} \wedge \text{purple})\} \\ 2 & \mapsto & \{\text{green}, \neg(\text{green} \wedge \text{purple})\} \\ 3 & \mapsto & \{\text{green}, \text{purple}, \text{green} \wedge \text{purple}\} \end{array}$$

Model checking CTL

Definition

The *formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists\bigcirc f \mid \exists(f \, \mathrm{U} \, f) \mid \forall\bigcirc f \mid \forall(f \, \mathrm{U} \, f)$$

Question

What is $Sat(\exists\bigcirc f)$?

Model checking CTL

Definition

The *formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists \bigcirc f \mid \exists (f \; \mathrm{U} \; f) \mid \forall \bigcirc f \mid \forall (f \; \mathrm{U} \; f)$$

Question

What is $Sat(\exists \bigcirc f)$?

Answer

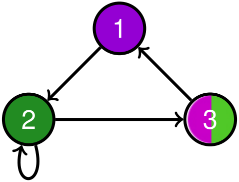
$Sat(\exists \bigcirc f) = \{\, s \in S \mid Post(s) \cap Sat(f) \neq \emptyset \,\}$ where
$Post(s) = \{\, s' \in S \mid s \rightarrow s' \,\}$.

Alternative view

Labels those states, that have a direct successor labelled with f, also with $\exists \bigcirc f$.

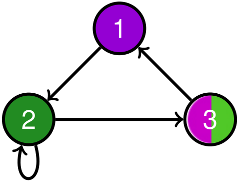
Example

$\exists \bigcirc$green



Example

$\exists\bigcirc$green



$$
\begin{array}{lll}
1 & \mapsto & \{\exists\bigcirc \text{green}\} \\
2 & \mapsto & \{\text{green}, \exists\bigcirc \text{green}\} \\
3 & \mapsto & \{\text{green}\}
\end{array}
$$

Model checking CTL

Definition

The *formulas* are defined by

$$f ::= a \mid f \wedge f \mid \neg f \mid \exists\bigcirc f \mid \exists(f \; \mathsf{U} \; f) \mid \forall\bigcirc f \mid \forall(f \; \mathsf{U} \; f)$$

Question

What is $Sat(\exists(f_1 \; \mathsf{U} \; f_2))$?

$s \in Sat(\exists(f_1 \; \mathsf{U} \; f_2))$

iff $\quad s \models \exists(f_1 \; \mathsf{U} \; f_2)$

iff $\quad s \models f_2 \vee (s \models f_1 \wedge \exists s \rightarrow t : t \models \exists(f_1 \; \mathsf{U} \; f_2))$

iff $\quad s \in Sat(f_2) \vee (s \in Sat(f_1) \wedge \exists t \in Post(s) : t \in Sat(\exists(f_1 \; \mathsf{U} \; f_2))$

iff $\quad s \in Sat(f_2) \cup \{ s \in Sat(f_1) \mid Post(s) \cap Sat(\exists(f_1 \; \mathsf{U} \; f_2)) \neq \emptyset\}$

Model checking CTL

$s \in Sat(\exists(f_1 \mathbin{\mathrm{U}} f_2))$

iff $s \models \exists(f_1 \mathbin{\mathrm{U}} f_2)$

iff $s \models f_2 \vee (s \models f_1 \wedge \exists s \to t : t \models \exists(f_1 \mathbin{\mathrm{U}} f_2))$

iff $s \in Sat(f_2) \vee (s \in Sat(f_1) \wedge \exists t \in Post(s) : t \in Sat(\exists(f_1 \mathbin{\mathrm{U}} f_2))$

iff $s \in Sat(f_2) \cup \{ s \in Sat(f_1) \mid Post(s) \cap Sat(\exists(f_1 \mathbin{\mathrm{U}} f_2)) \neq \emptyset\}$

Proposition

$Sat(\exists(f_1 \mathbin{\mathrm{U}} f_2))$ is the smallest subset T of S such that

$$T = Sat(f_2) \cup \{ s \in Sat(f_1) \mid Post(s) \cap T \neq \emptyset\}.$$

Model checking CTL

$s \in Sat(\exists(f_1 \text{ U } f_2))$

iff $s \models \exists(f_1 \text{ U } f_2)$

iff $s \models f_2 \vee (s \models f_1 \wedge \exists s \to t : t \models \exists(f_1 \text{ U } f_2))$

iff $s \in Sat(f_2) \vee (s \in Sat(f_1) \wedge \exists t \in Post(s) : t \in Sat(\exists(f_1 \text{ U } f_2))$

iff $s \in Sat(f_2) \cup \{ s \in Sat(f_1) \mid Post(s) \cap Sat(\exists(f_1 \text{ U } f_2)) \neq \emptyset\}$

Proposition

$Sat(\exists(f_1 \text{ U } f_2))$ is the smallest subset T of S such that

$$T = Sat(f_2) \cup \{ s \in Sat(f_1) \mid Post(s) \cap T \neq \emptyset\}.$$

Question

Does such a smallest subset exist?

Smallest Subset

Definition

The function $F : 2^S \to 2^S$ is defined by

$$F(T) = Sat(f_2) \cup \{ s \in Sat(f_1) \mid Post(s) \cap T \neq \emptyset \}.$$

Definition

A function $G : 2^S \to 2^S$ is monotone if for all $T, U \in 2^S$, if $T \subseteq U$ then $G(T) \subseteq G(U)$.

Smallest Subset

Proposition

F is monotone.

Proof

Let $T, U \in 2^S$. Assume that $T \subseteq U$. Let $s \in F(T)$. It remains to prove that $s \in F(U)$. Then $s \in Sat(f_2)$ or $s \in Sat(f_1)$ and $Post(s) \cap T \neq \emptyset$. We distinguish two cases.

- If $s \in Sat(f_2)$ then $s \in F(U)$.
- If $s \in Sat(f_1)$ and $Post(s) \cap T \neq \emptyset$ then $Post(s) \cap U \neq \emptyset$ since $T \subseteq U$. Hence, $s \in F(U)$.

Smallest Subset

Definition

For each $n \in \mathbb{N}$, the set F_n is defined by

$$F_n = \begin{cases} \emptyset & \text{if } n = 0 \\ F(F_{n-1}) & \text{otherwise} \end{cases}$$

Smallest Subset

Proposition

For all $n \in \mathbb{N}$, $F_n \subseteq F_{n+1}$.

Proof

We prove this by induction on n. In the base case, $n = 0$, we have that

$$F_0 = \emptyset \subseteq F_1.$$

In the inductive case, we have $n > 1$. By induction, $F_{n-1} \subseteq F_n$. Since F is monotone, we have that

$$F_n = F(F_{n-1}) \subseteq F(F_n) = F_{n+1}.$$

Smallest Subset

Proposition

If S is a finite set. then $F_n = F_{n+1}$ for some $n \in \mathbb{N}$.

Proof

Suppose that S contains m elements. Towards a contradiction, assume that $F_n \neq F_{n+1}$ for all $n \in \mathbb{N}$. Then $F_n \subset F_{n+1}$ for all $n \in \mathbb{N}$. Hence, F_n contains at least n elements. Therefore, F_{m+1} contains more elements than S. This contradicts that $F_{m+1} \subseteq S$.

We denote the F_n with $F_n = F_{n+1}$ by $\mathit{fix}(F)$.

Smallest Subset

Proposition

For all $T \subseteq S$, if $F(T) = T$ then $fix(F) \subseteq T$.

Proof

First, we prove that for all $n \in \mathbb{N}$, $F_n \subseteq T$ by induction on n. In the base case, $n = 0$, we have that

$$F_0 = \emptyset \subseteq T.$$

In the inductive case, we have $n > 1$. By induction, $F_{n-1} \subseteq T$. By induction

$$F_n = F(F_{n-1}) \subseteq F(T) = T.$$

Since $fix(F) = F_n$ for some $n \in \mathbb{N}$, we can conclude that $fix(F) \subseteq T$.

Smallest Subset

Corollary

fix(F) is the smallest T of S such that $F(T) = T$.