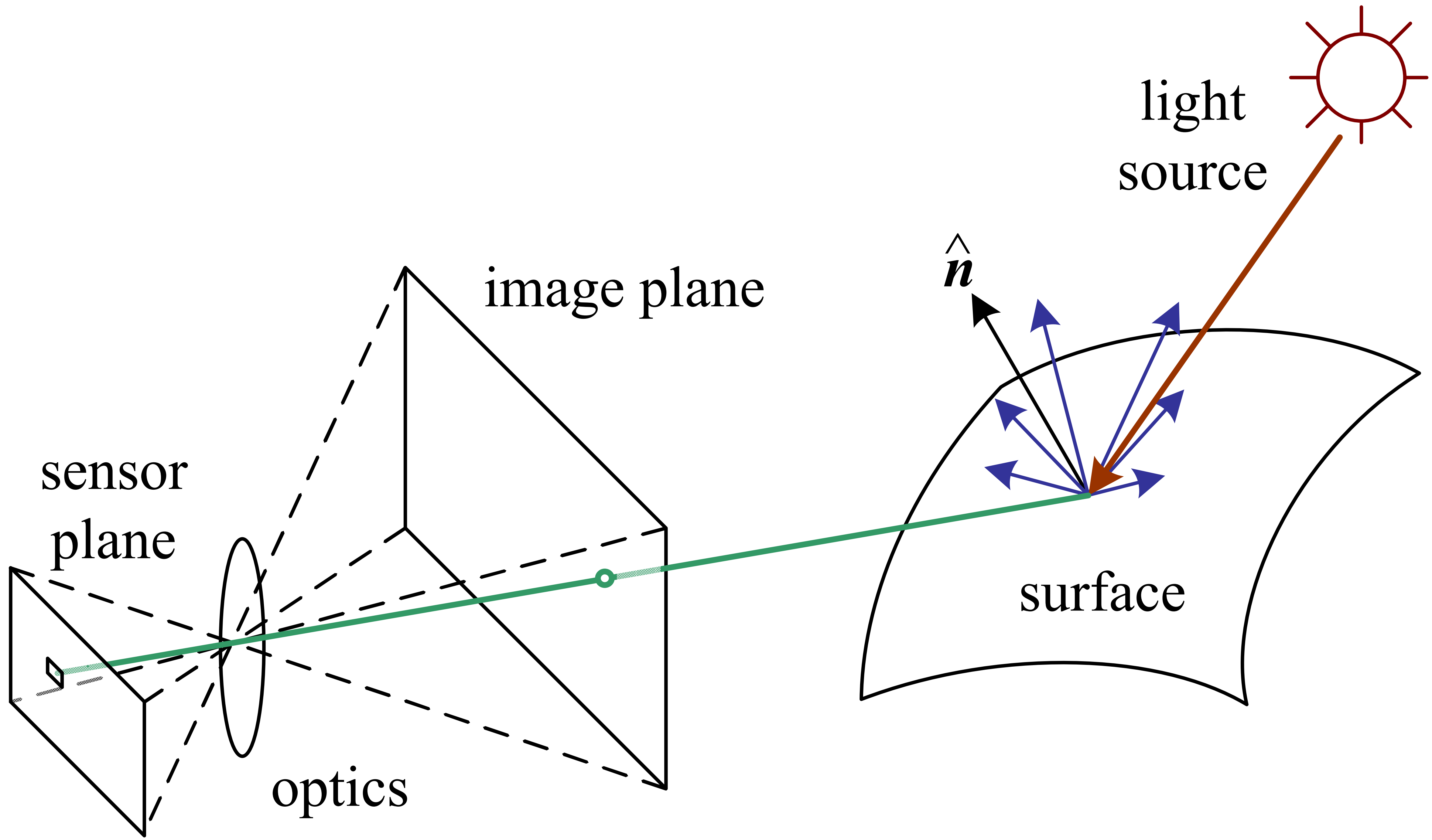
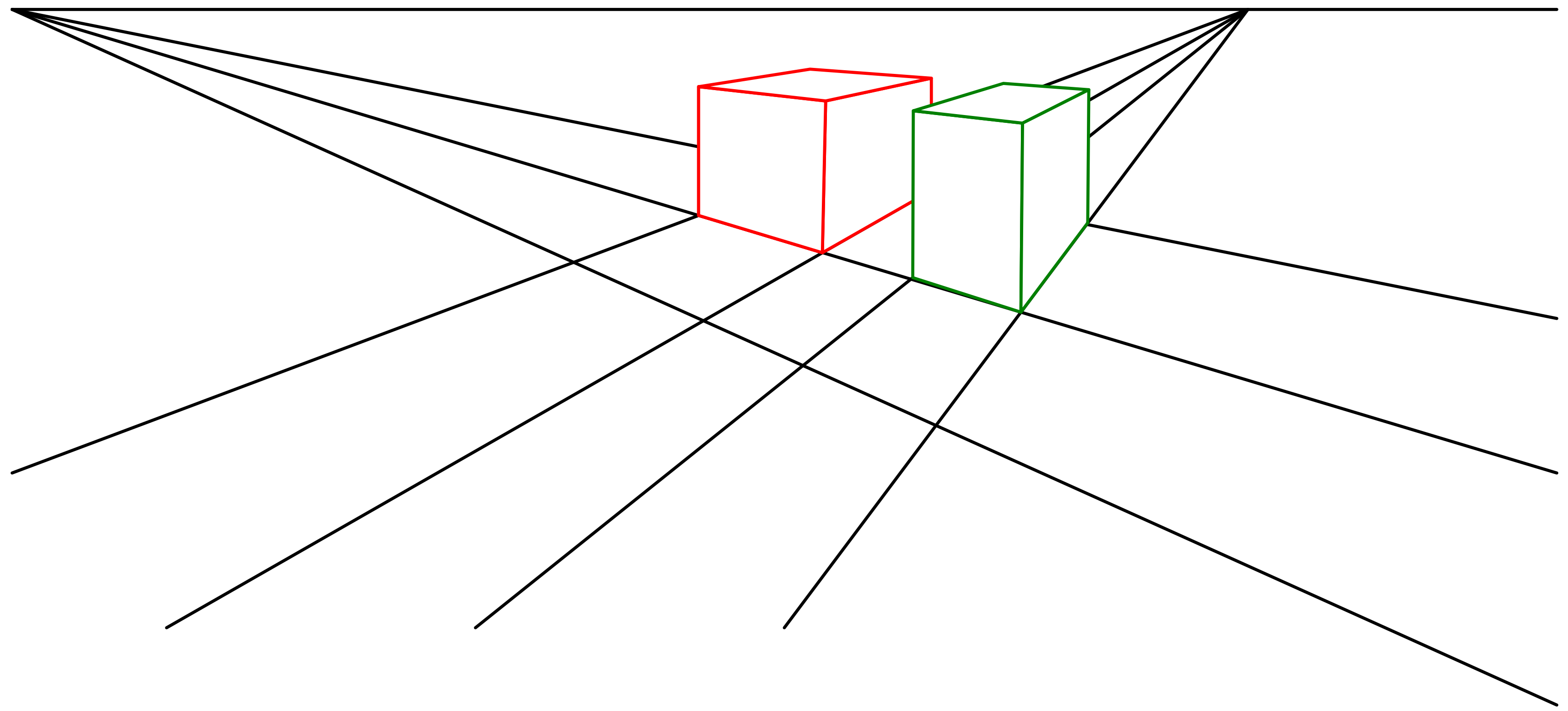
2. Image Formation



2.1 Geometric Primitives & Transformations



Outline

- Geometric primitives
- 2D transformations
- 3D transformations
- 3D rotations
- 3D to 2D projections
- Lens Distortions

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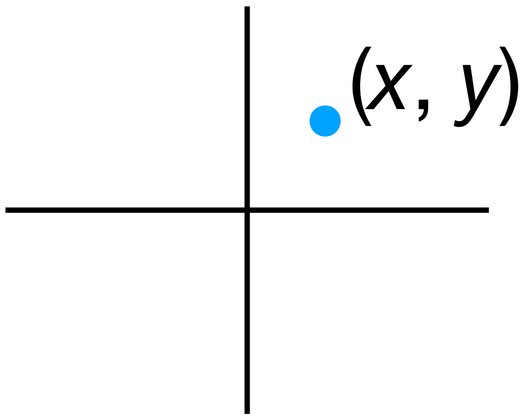
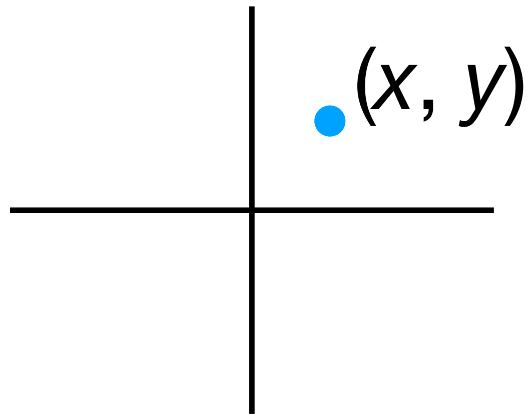
2D Points



2D Points

2D point (e.g., a pixel coordinate in an image):



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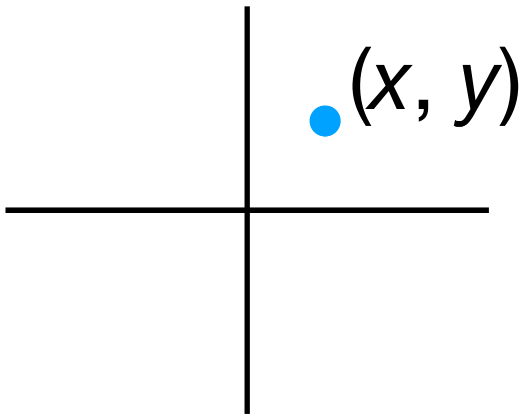

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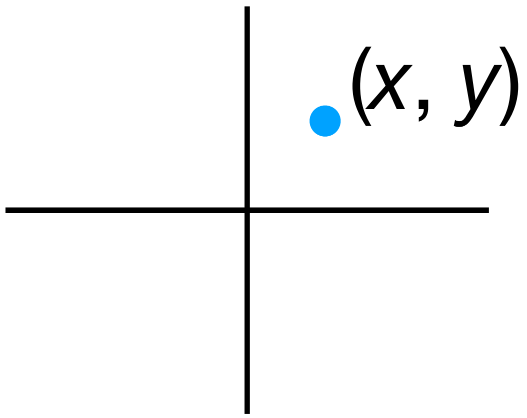

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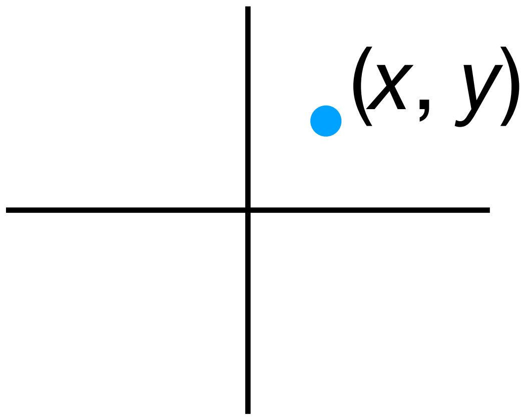

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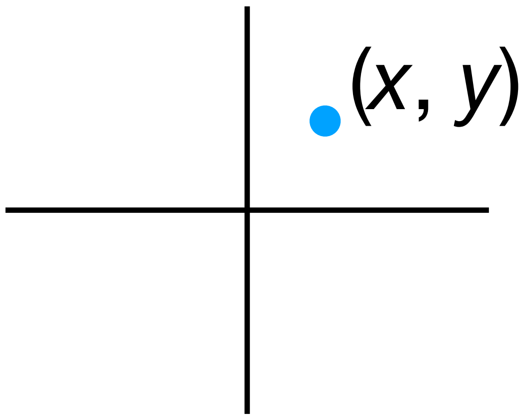

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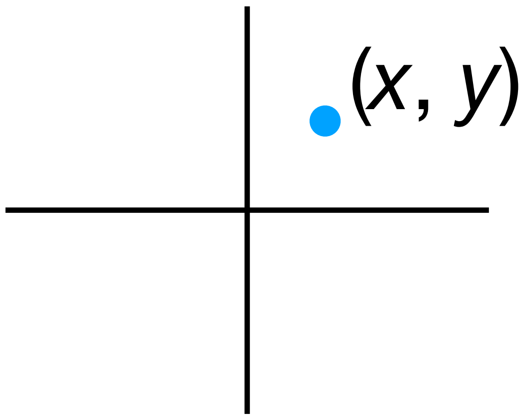

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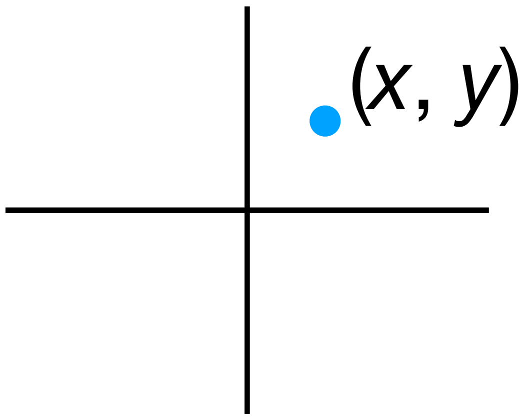

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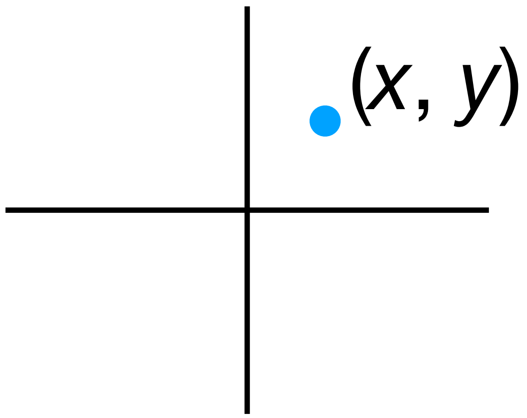

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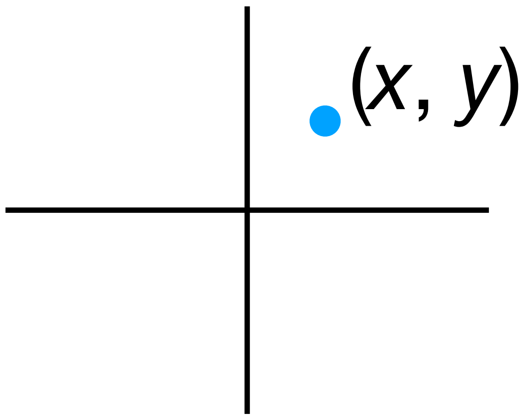

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$s\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{x}} \;\forall s \in \mathcal{R}$

Augmented Vectors

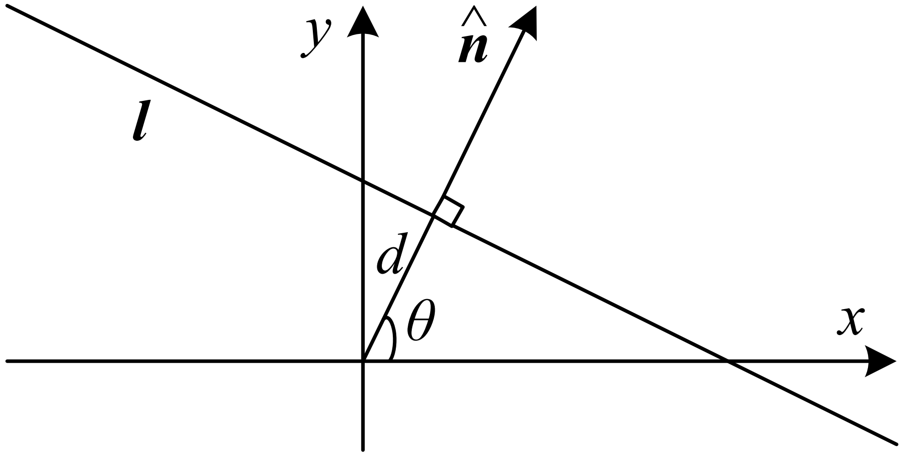
A homogenous vector can be converted back to an *inhomogeneous* vector by dividing by the last element:

$$\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}, \tilde{w}) = \tilde{w}(x, y, 1) = \tilde{w}\bar{\boldsymbol{x}}$$

Homogeneous vector

Augmented vector

2D Lines



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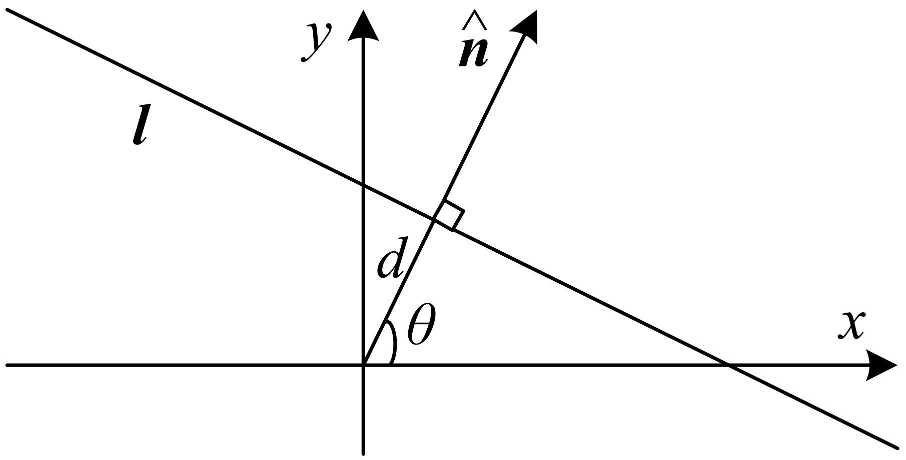

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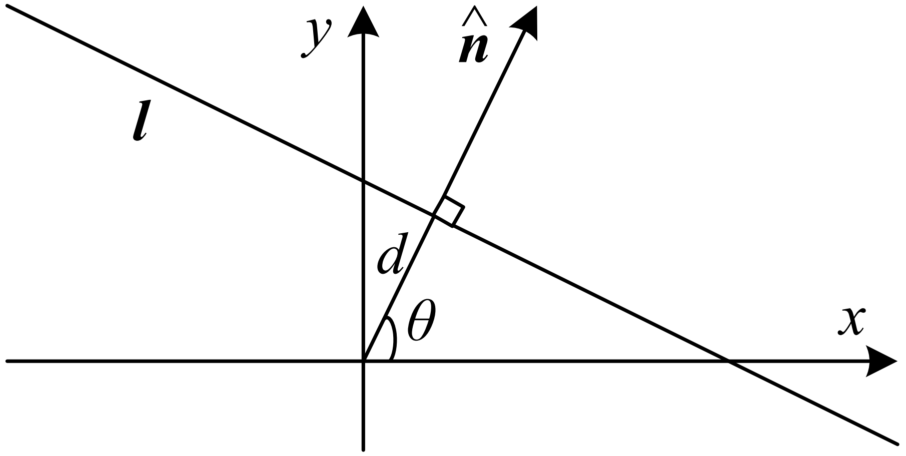

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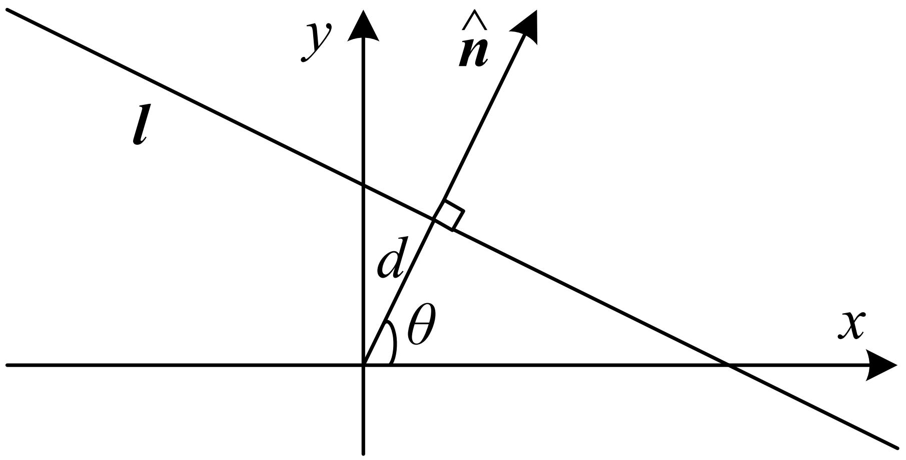

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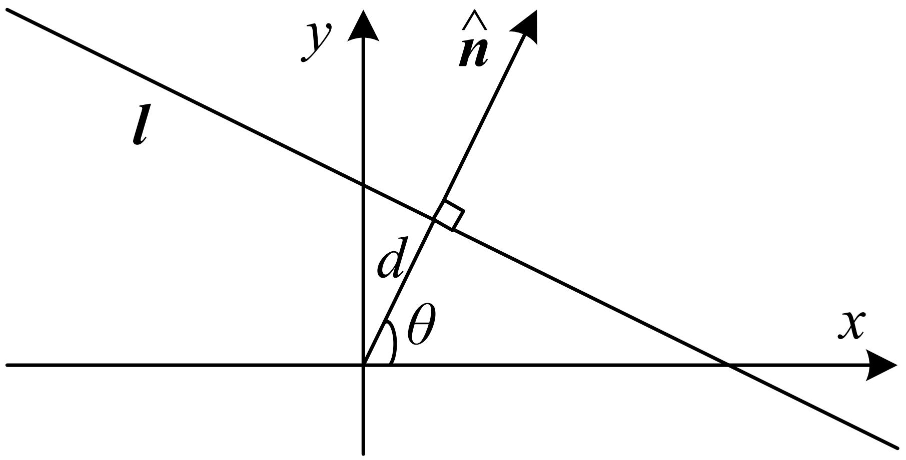
$\hat{\boldsymbol{n}}$ is the unit normal perpendicular to the line

$|d|$ is the distance of the line from the origin



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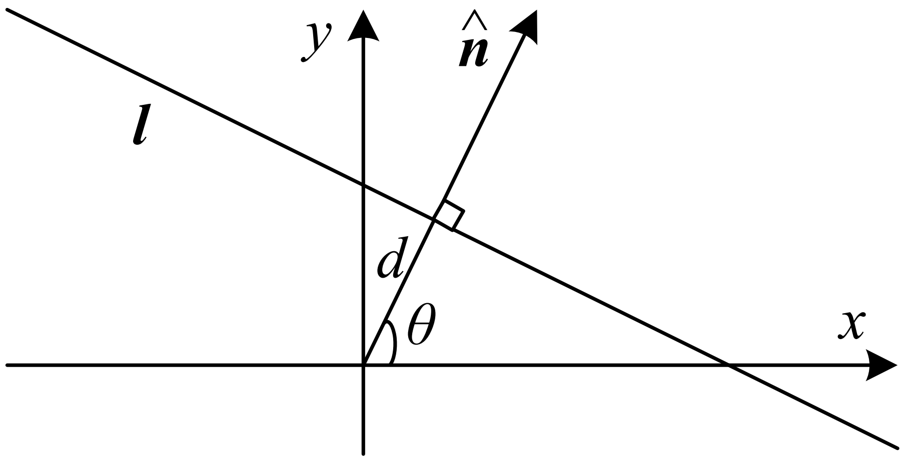

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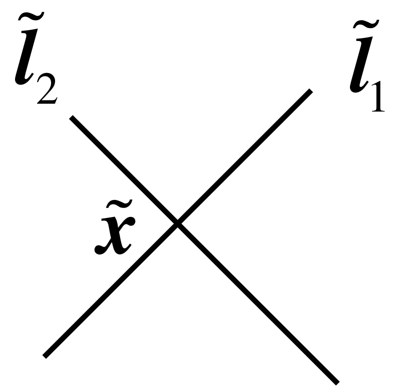

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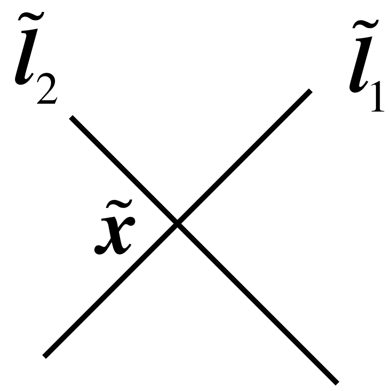

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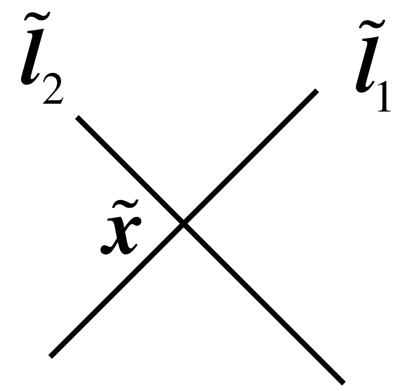

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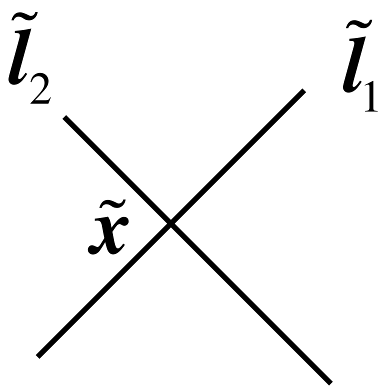

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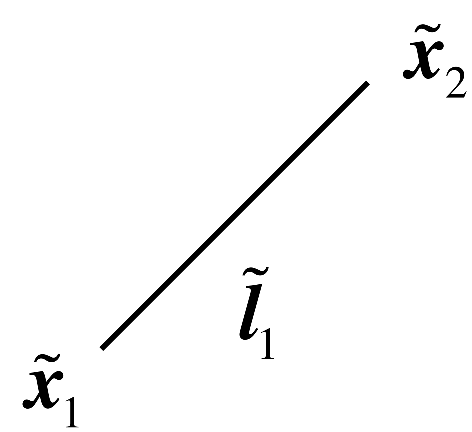

3D Points

Straightforward extension from 2D:

Inhomogeneous: $\boldsymbol{x} = (x, y, z) \in \mathcal{R}^3$

Homogeneous: $\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}) \in \mathcal{P}^3$

Augmented: $\bar{\boldsymbol{x}} = (x, y, z, 1)$

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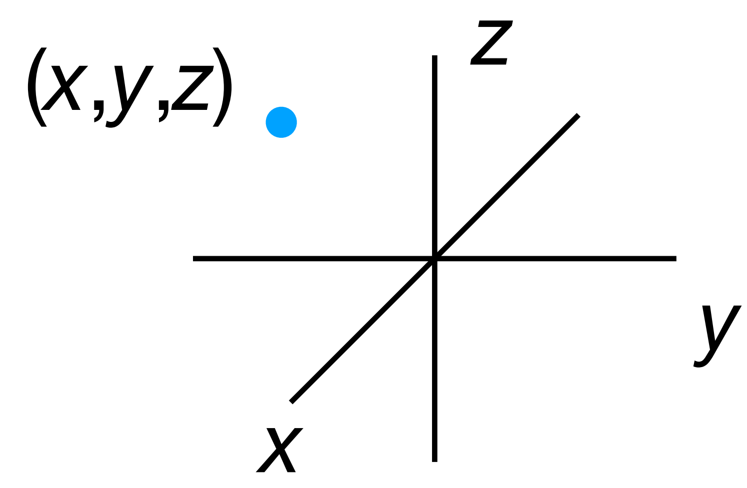
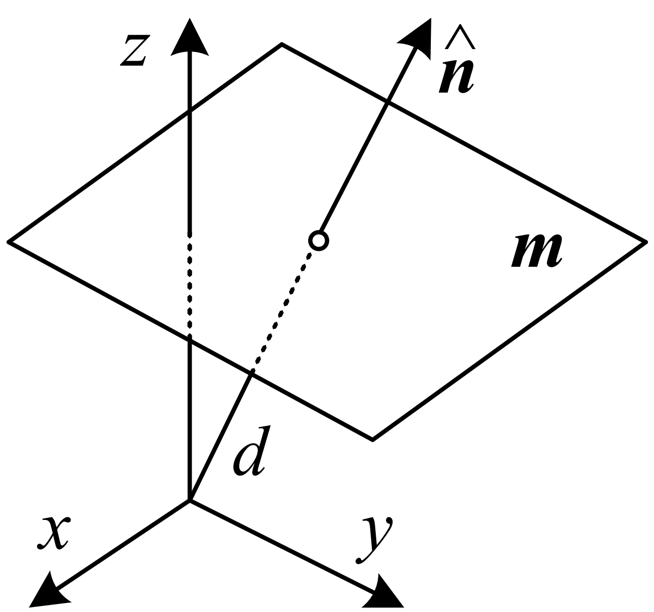


3D Planes



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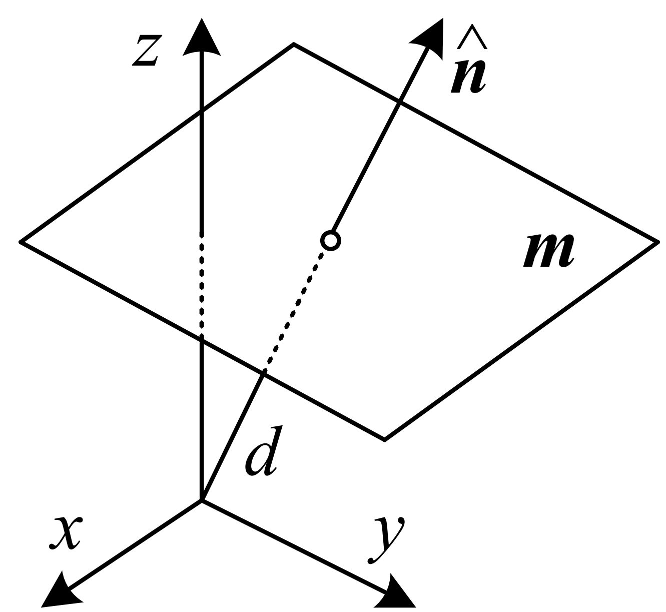

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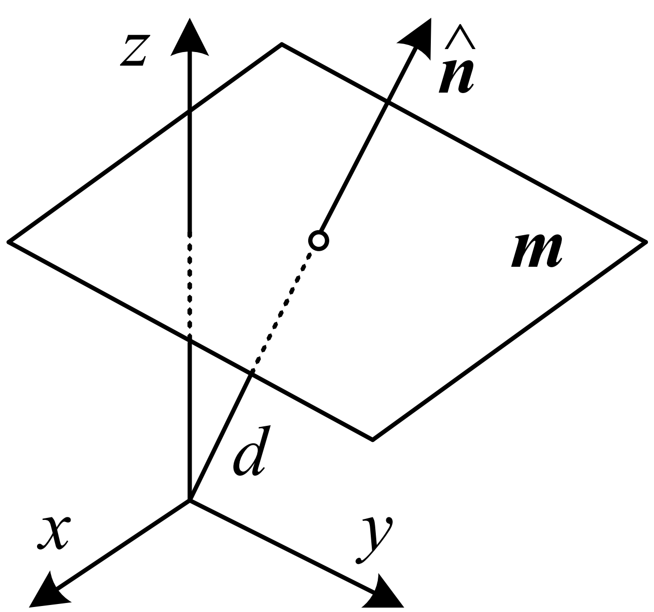

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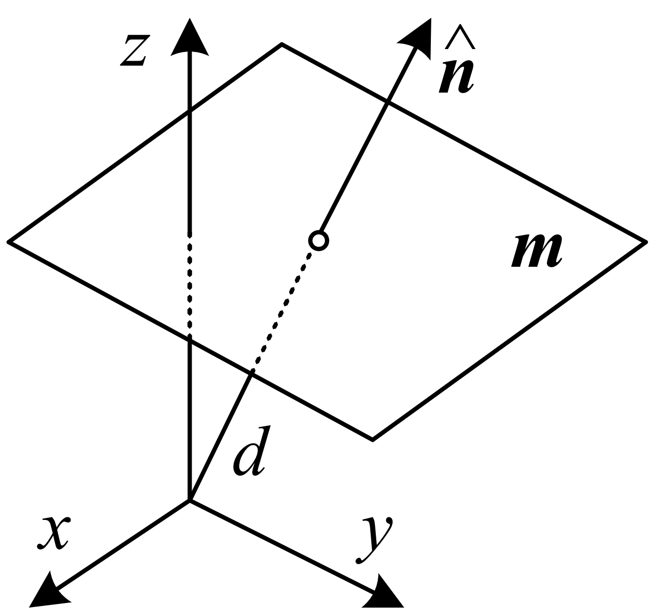

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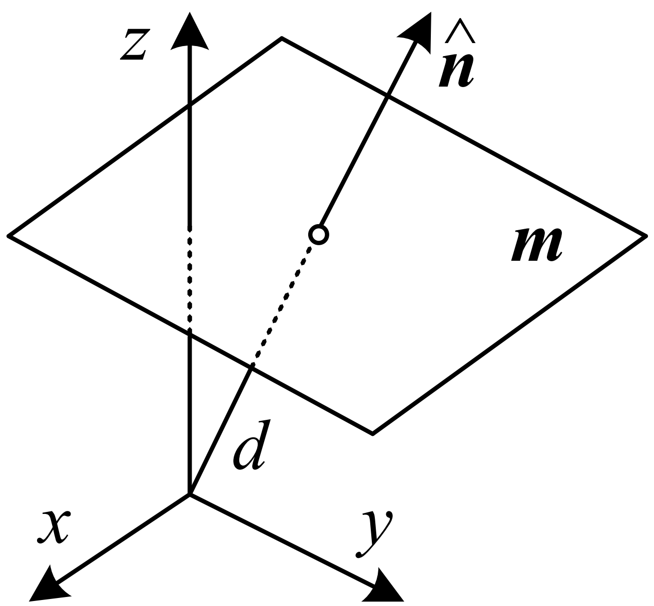

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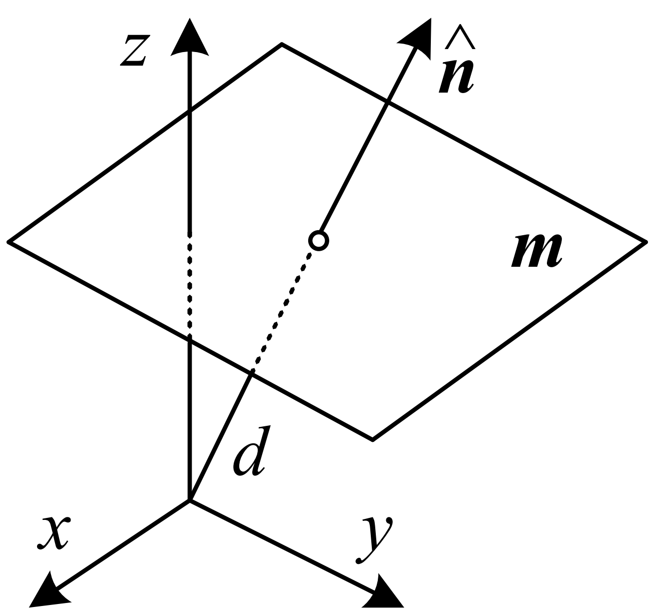
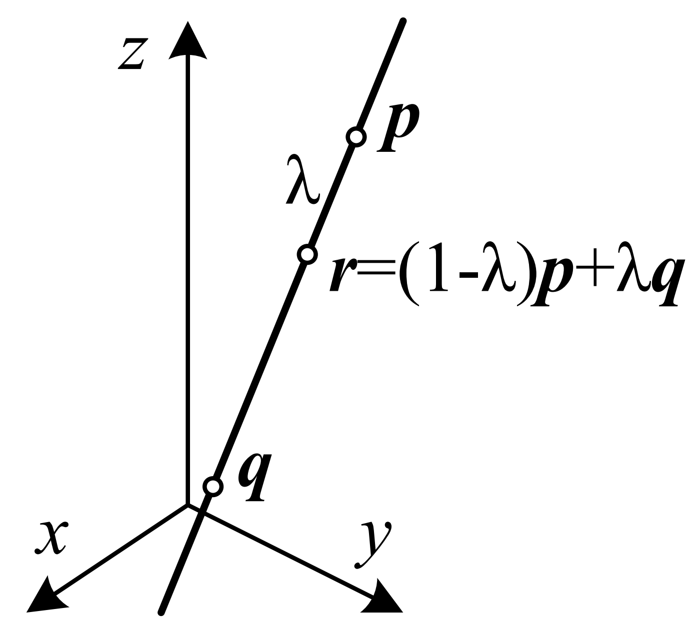
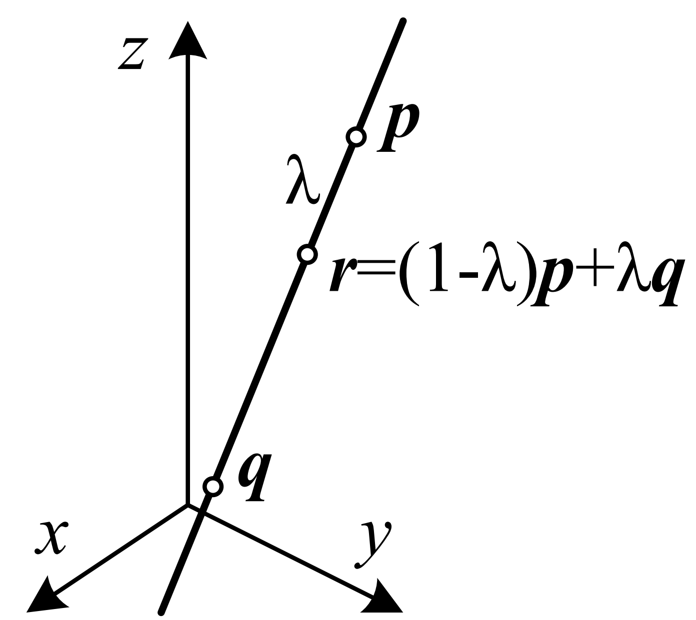


3D Lines



3D Lines

A 3D line can be represented using two points $\boldsymbol{p}$ and $\boldsymbol{q}$ that lie on the line.



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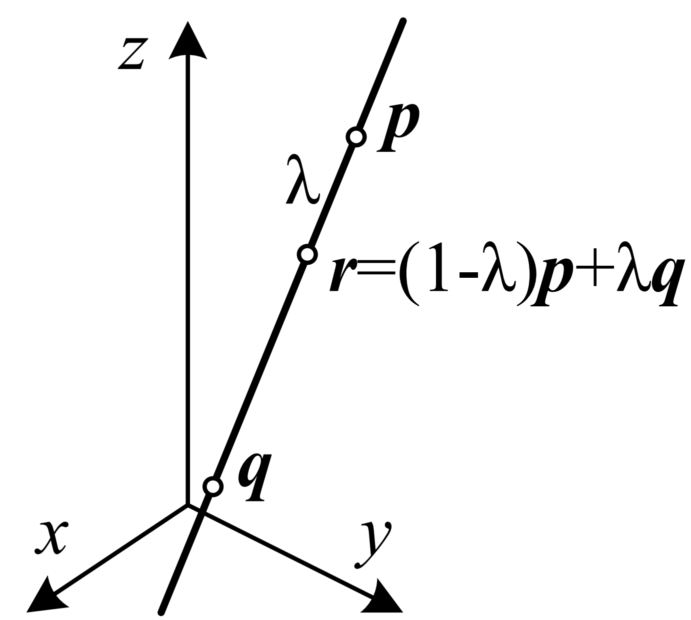

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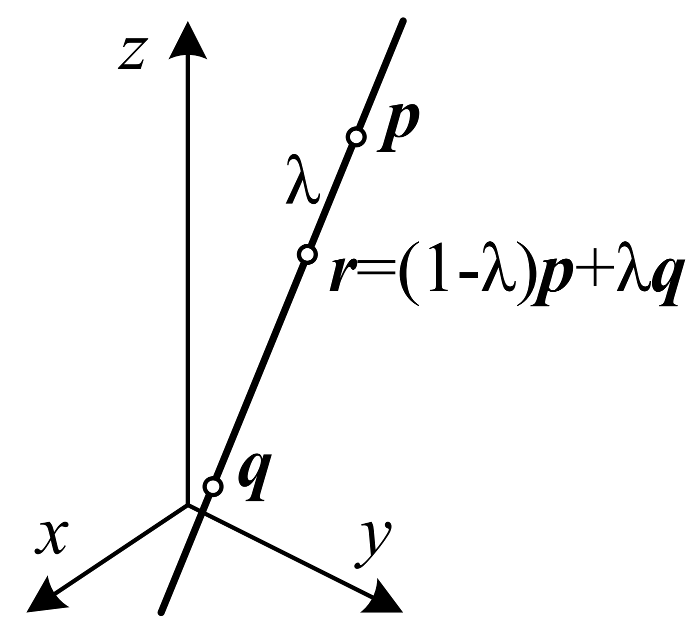

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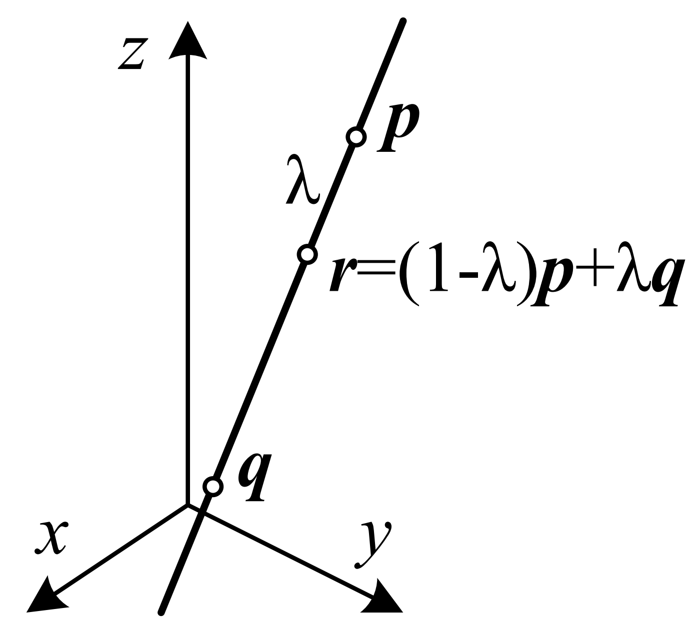

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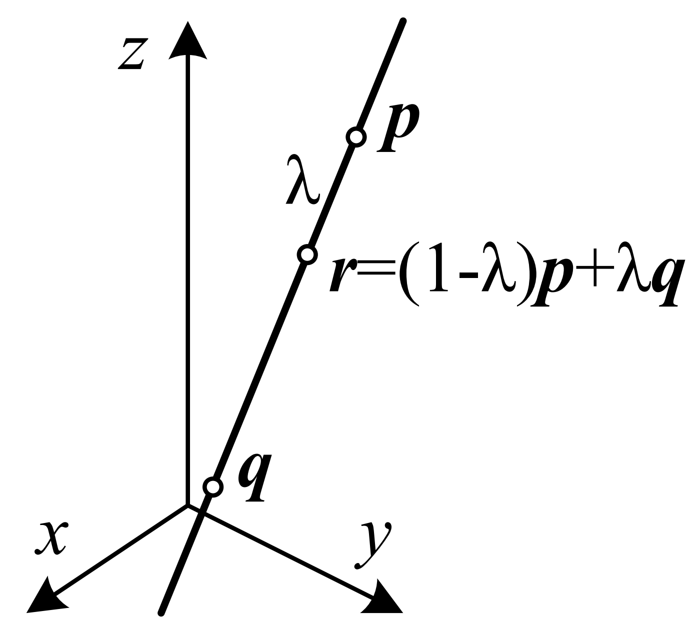

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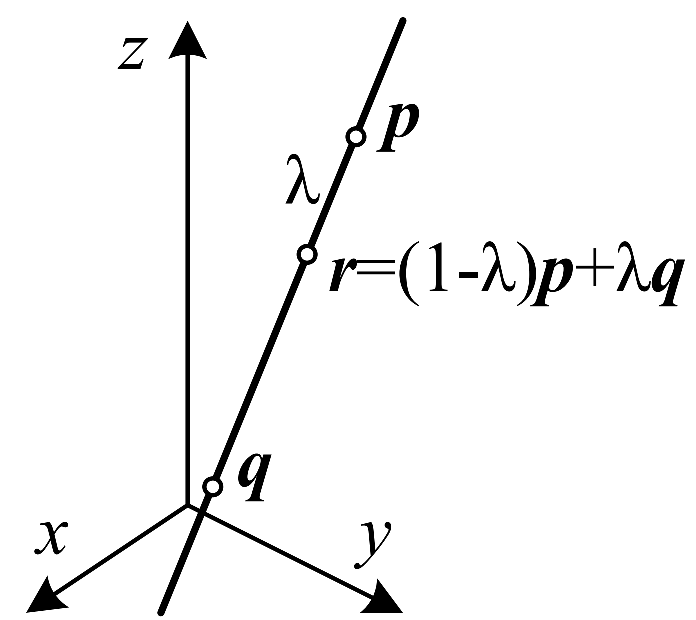

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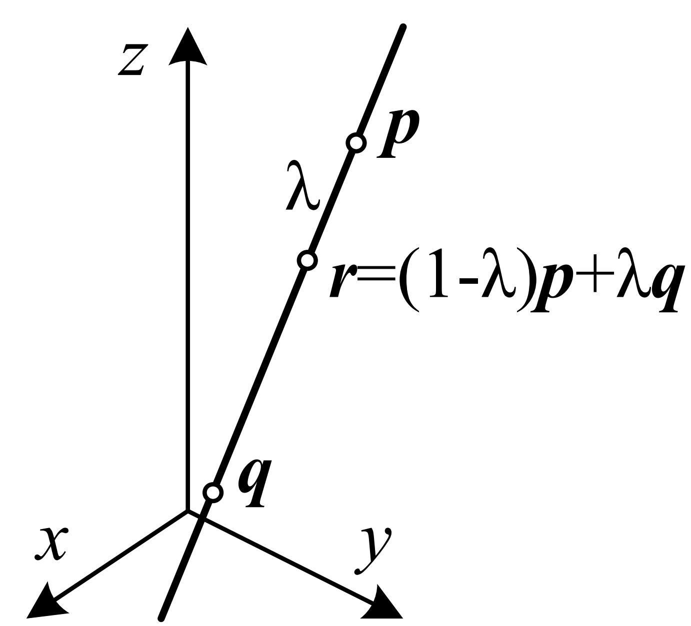

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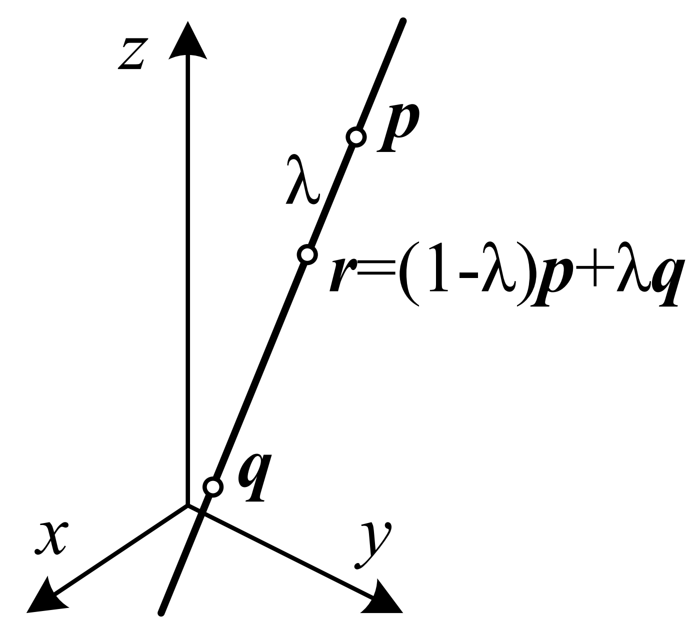

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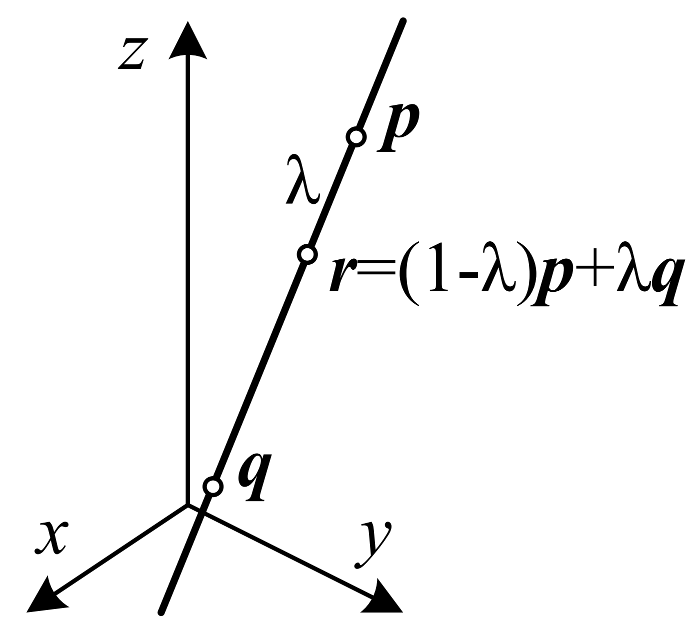
Then:

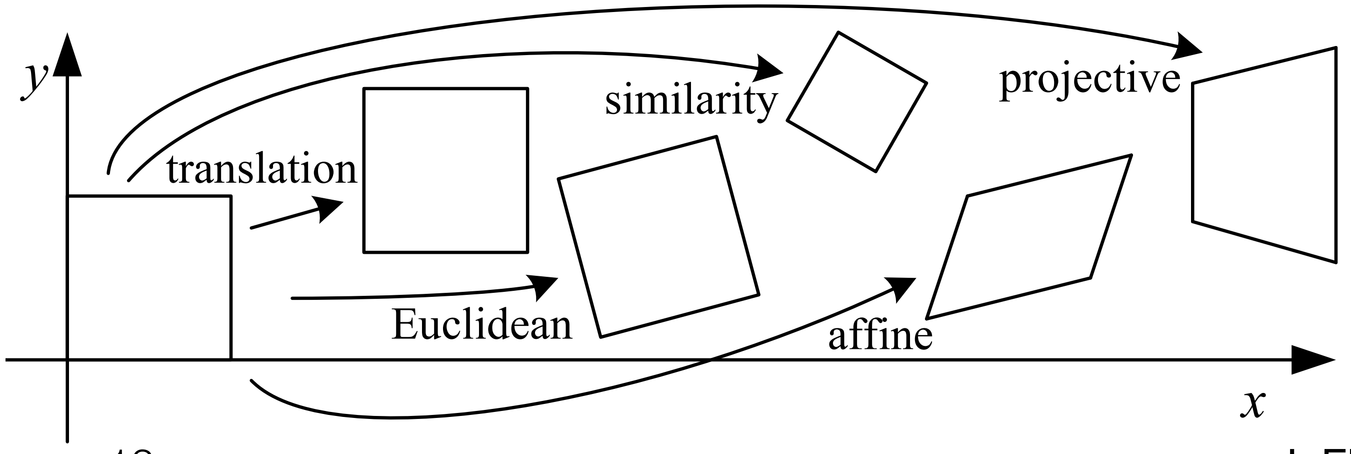
$$\boldsymbol{r} = \boldsymbol{p} + \lambda\hat{\boldsymbol{d}}$$



Outline

- ❖ Geometric primitives
- ❖ **2D transformations**
- ❖ 3D transformations
- ❖ 3D rotations
- ❖ 3D to 2D projections
- ❖ Lens Distortions

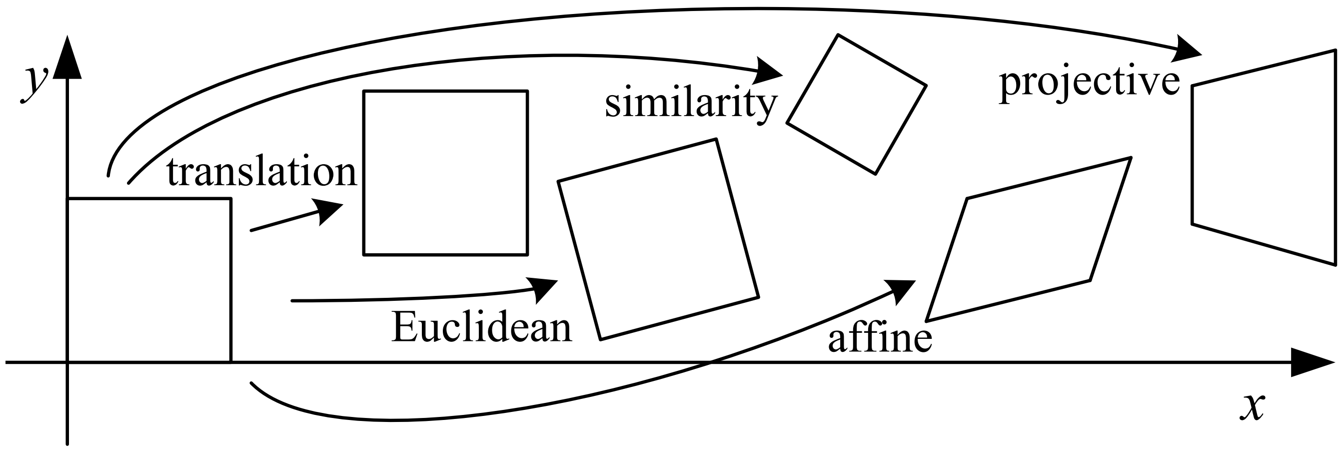
2D Translation



2D Translation

2D translations can be written as $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{t}$ or

$$\boldsymbol{x}' = \left[\begin{array}{cc} \boldsymbol{I} & \boldsymbol{t} \end{array} \right] \bar{\boldsymbol{x}}$$



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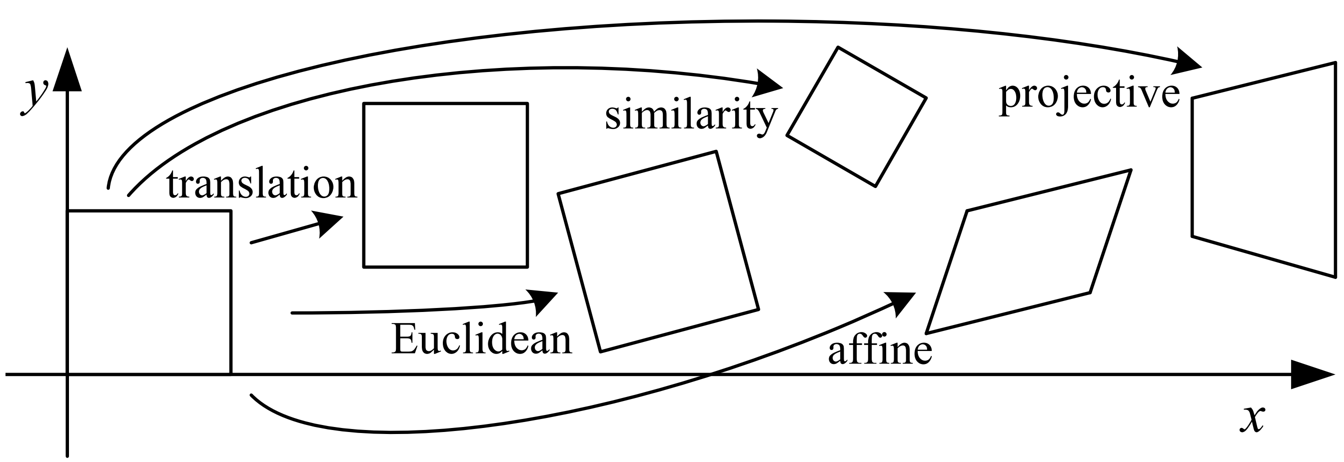
where $\boldsymbol{I}$ is the (2×2) identity matrix



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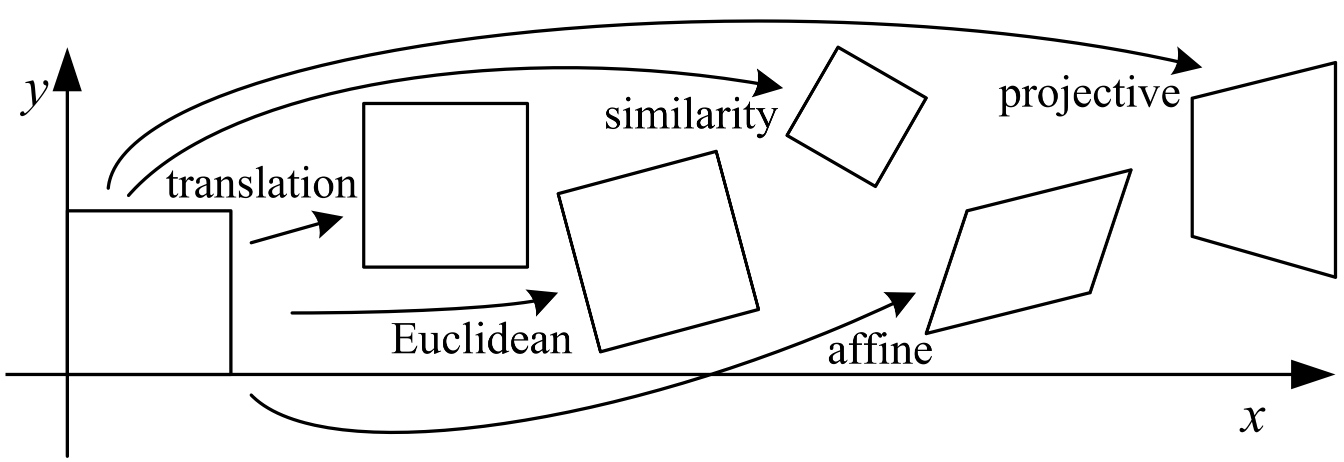

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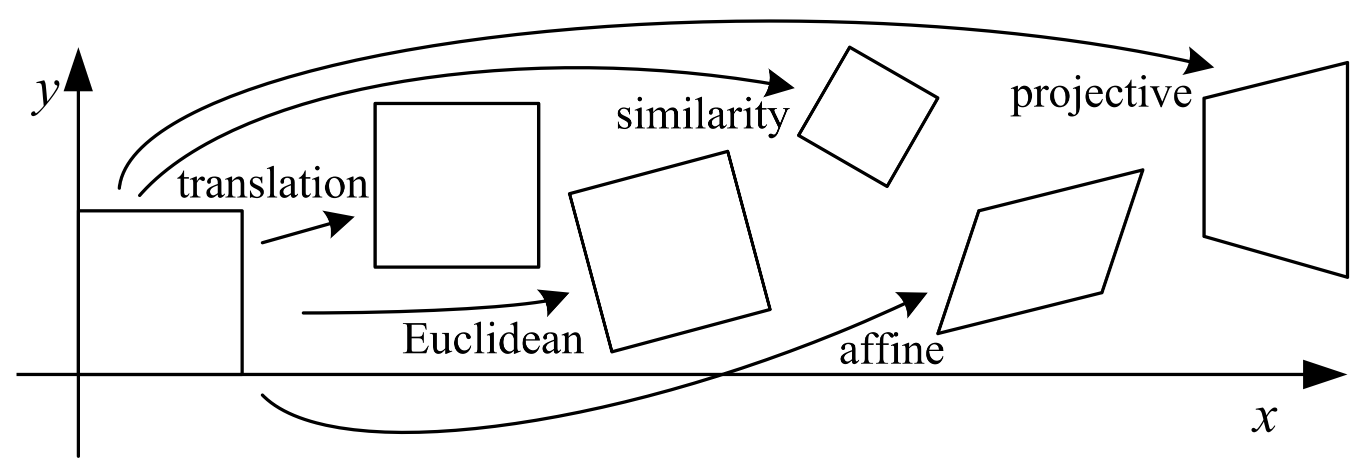
or

$$\bar{\boldsymbol{x}}' = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \bar{\boldsymbol{x}}$$



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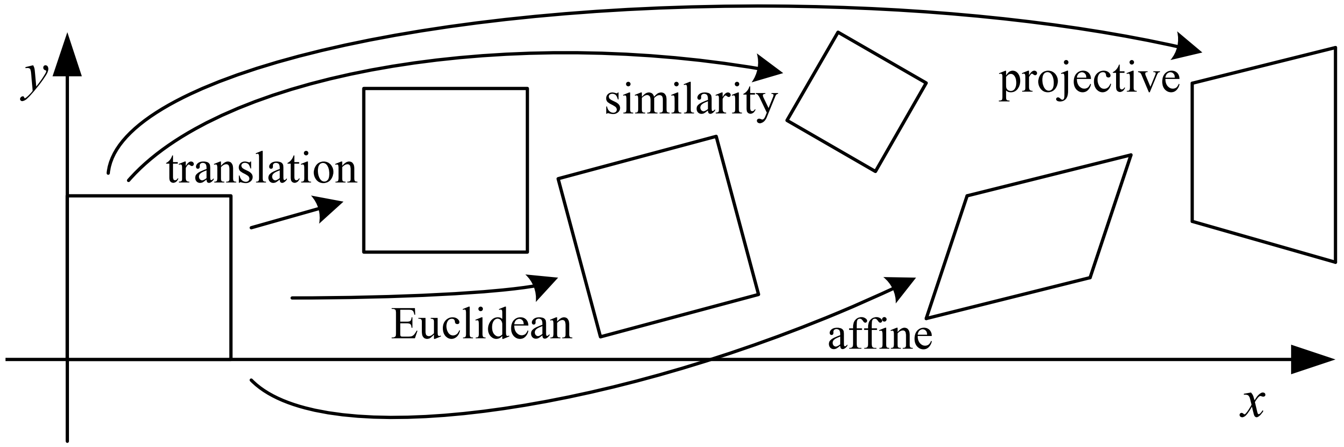
$$\bar{\boldsymbol{x}}' = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \bar{\boldsymbol{x}}$$

Note: Whenever an augmented vector appears on both sides, it can be replaced by a full homogenous vector.



Euclidean Transformation (2D Rotation +

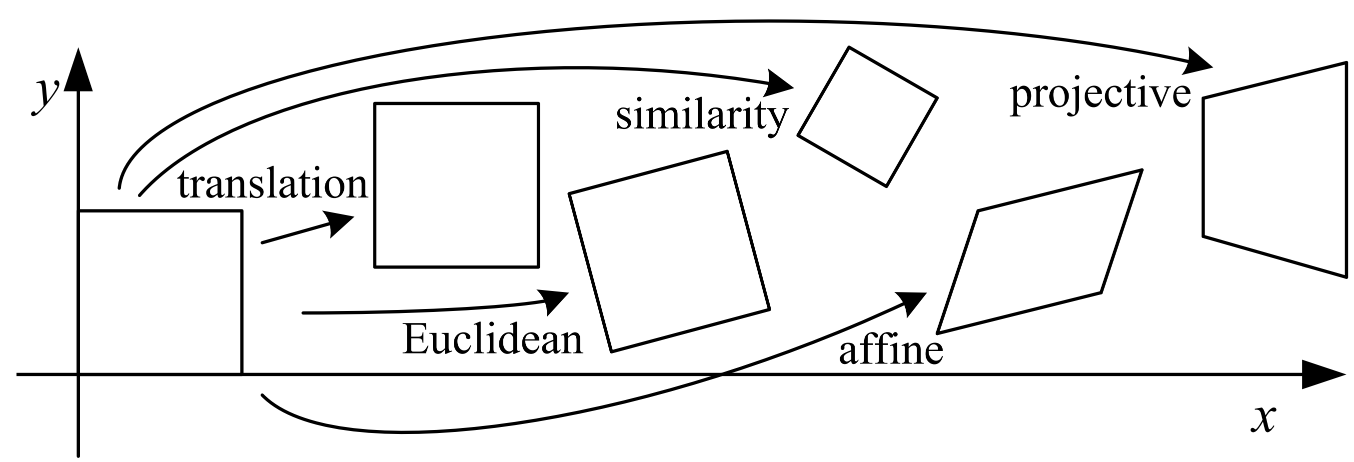
$$\boldsymbol{x}' = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t} \text{ or}$$

$$\boldsymbol{x}' = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \end{bmatrix} \bar{\boldsymbol{x}}$$

where

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is an orthonormal rotation matrix with $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$ and $|\boldsymbol{R}| = 1$.



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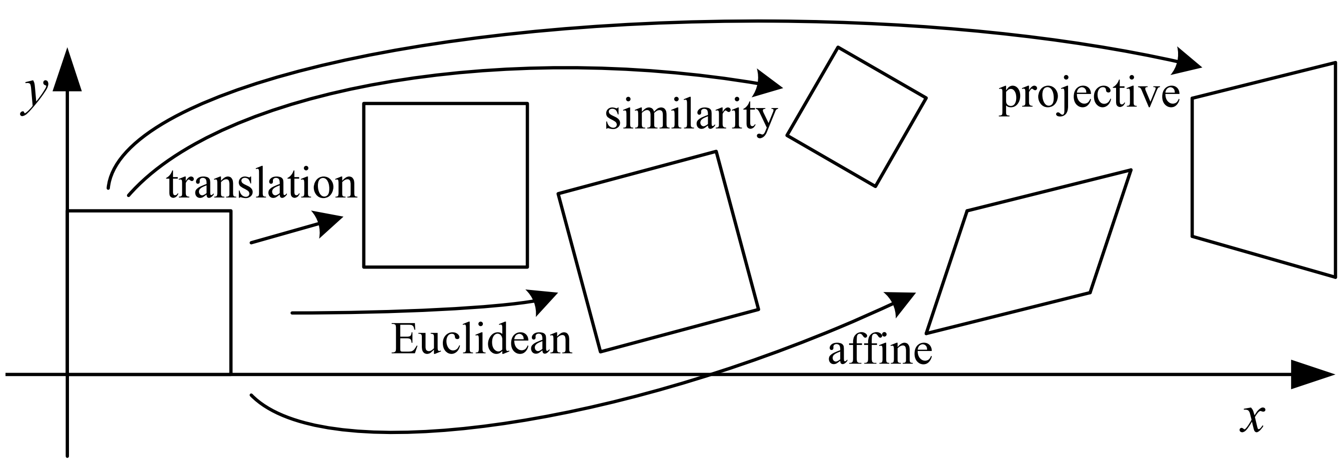
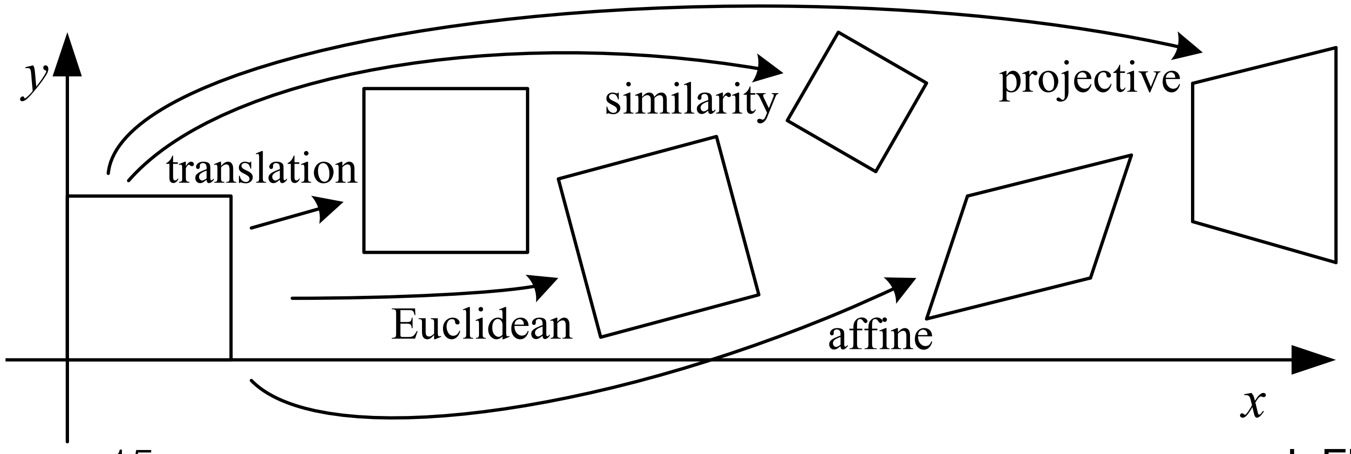
Preserves Euclidean distances

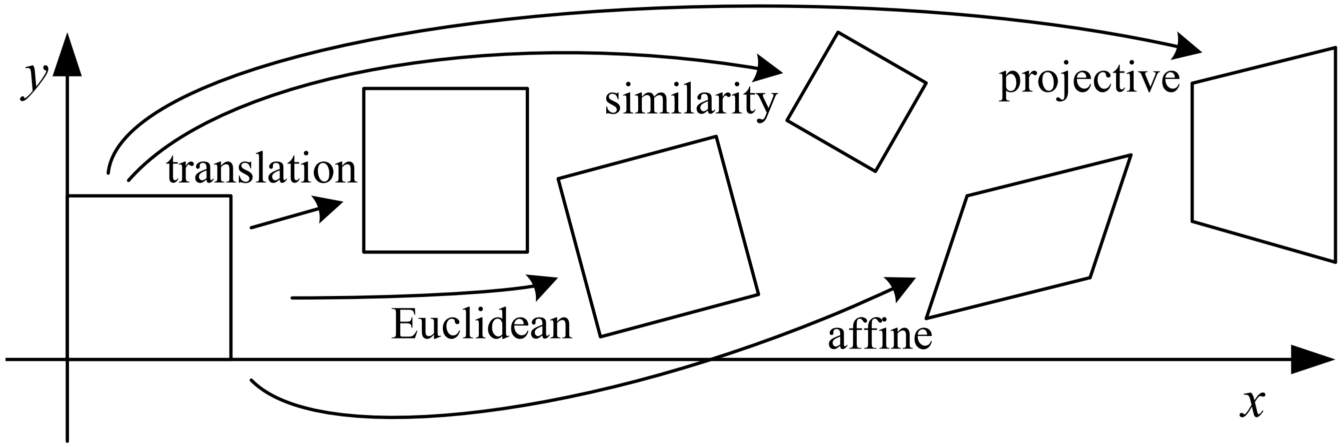


Similarity Transformation



Similarity Transformation

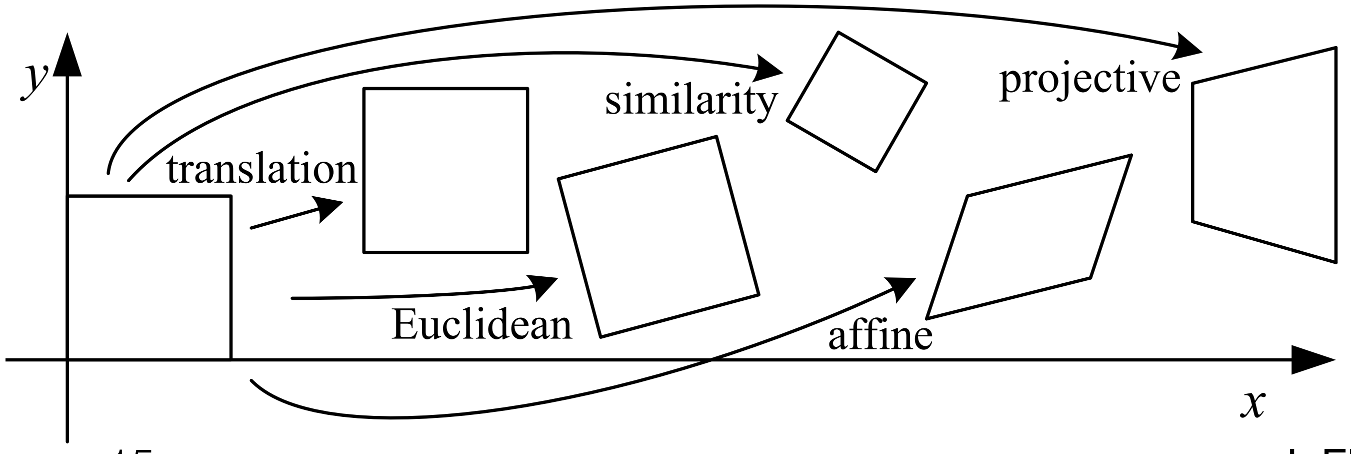
$$\boldsymbol{x}' = s\boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$$



Similarity Transformation

$$\boldsymbol{x}' = s\boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$$

$$\boldsymbol{x}' = \begin{bmatrix} s\boldsymbol{R} & \boldsymbol{t} \end{bmatrix} \bar{\boldsymbol{x}} = \begin{bmatrix} a & -b & t_x \\ b & a & t_y \end{bmatrix} \bar{\boldsymbol{x}}$$



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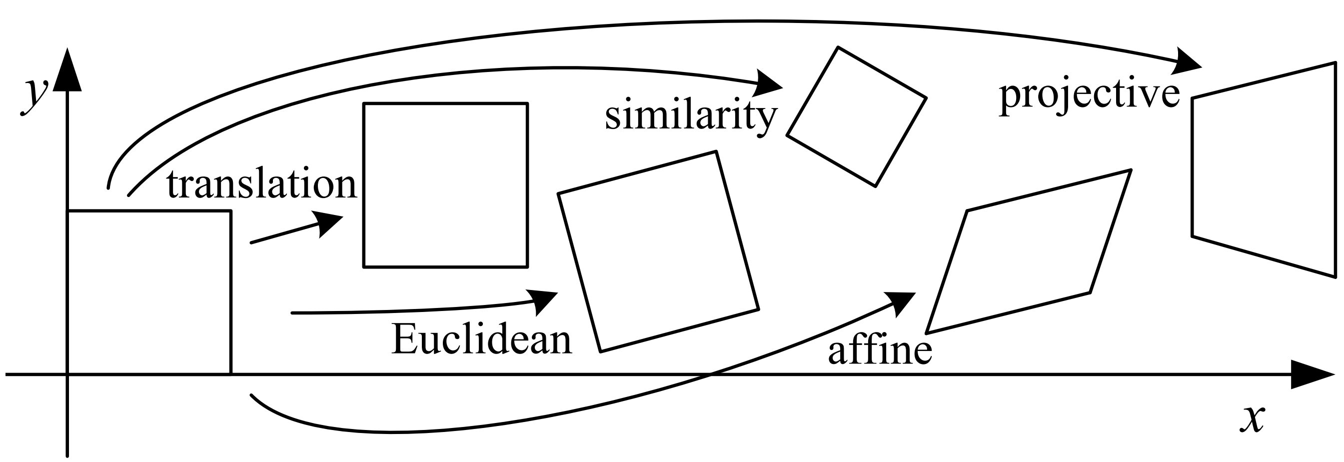
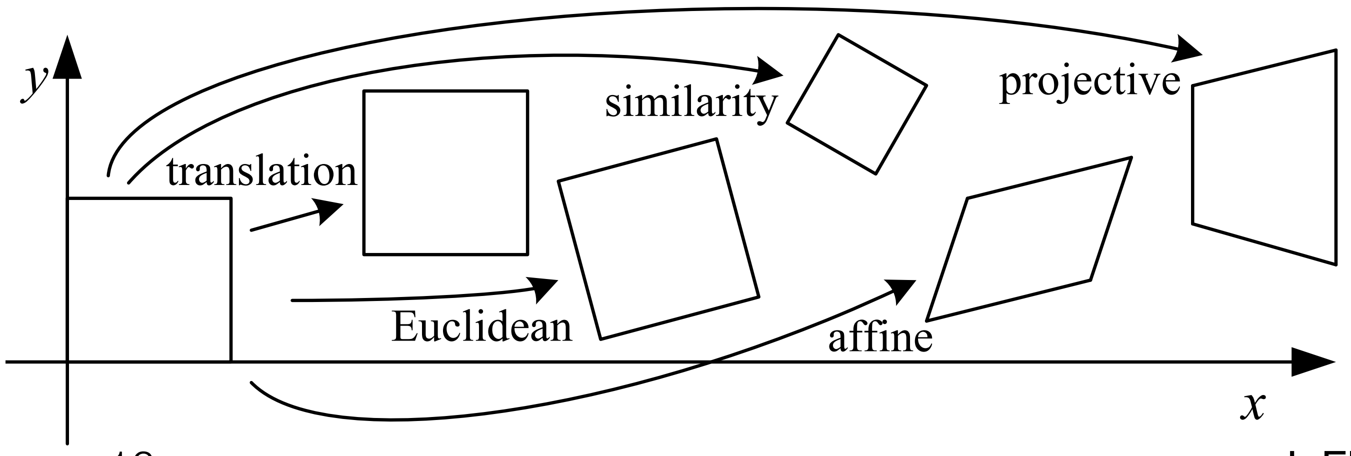


Affine Transformation

$\boldsymbol{x}' = \boldsymbol{A}\bar{\boldsymbol{x}}$, where $\boldsymbol{A}$ is an arbitrary 2×3 matrix

$$\boldsymbol{x}' = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \bar{\boldsymbol{x}}$$



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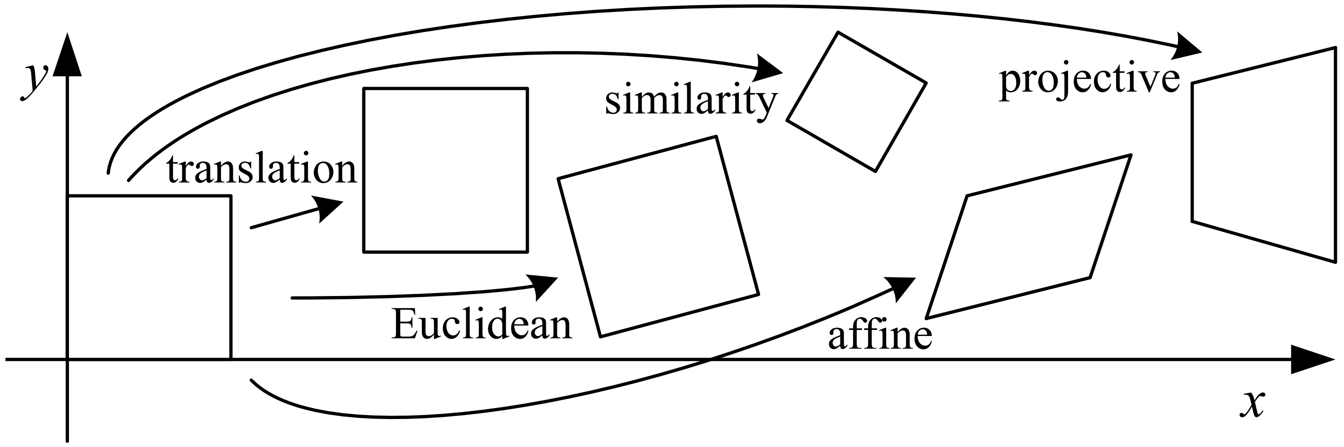
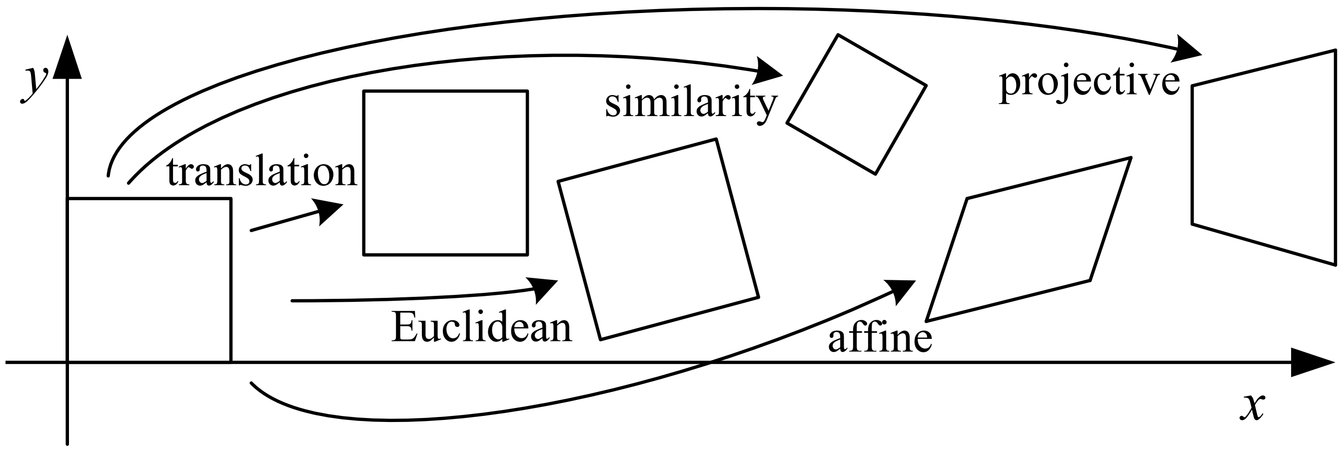
$$\boldsymbol{x}' = \boldsymbol{A}\bar{\boldsymbol{x}}, \text{ where } \boldsymbol{A} \text{ is an arbitrary } 2 \times 3 \text{ matrix}$$

$$\boldsymbol{x}' = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \bar{\boldsymbol{x}}$$

Preserves parallelism

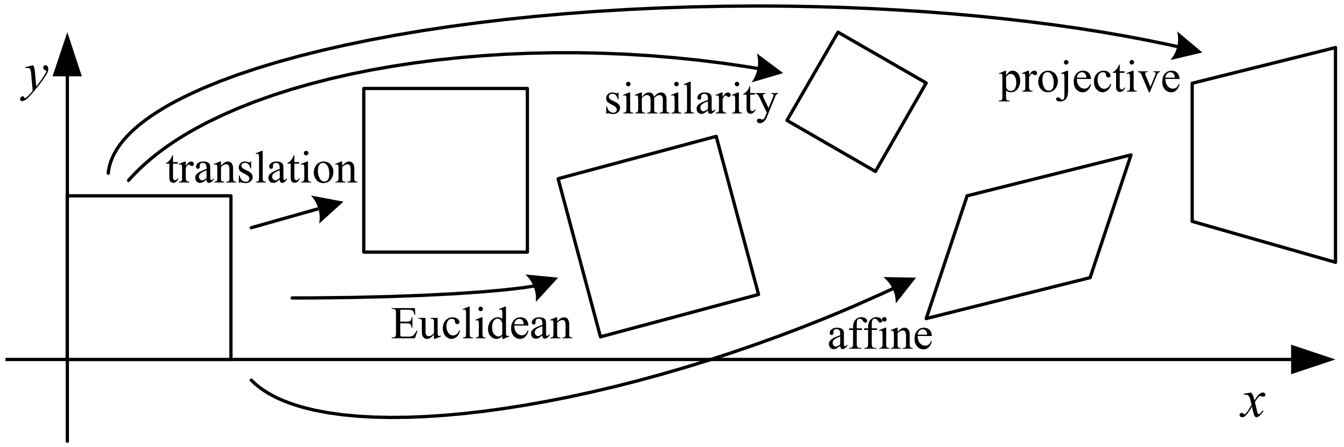


Projective Transformation (Homography)



Projective Transformation (Homography)

$$\tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{H}}\tilde{\boldsymbol{x}}$$



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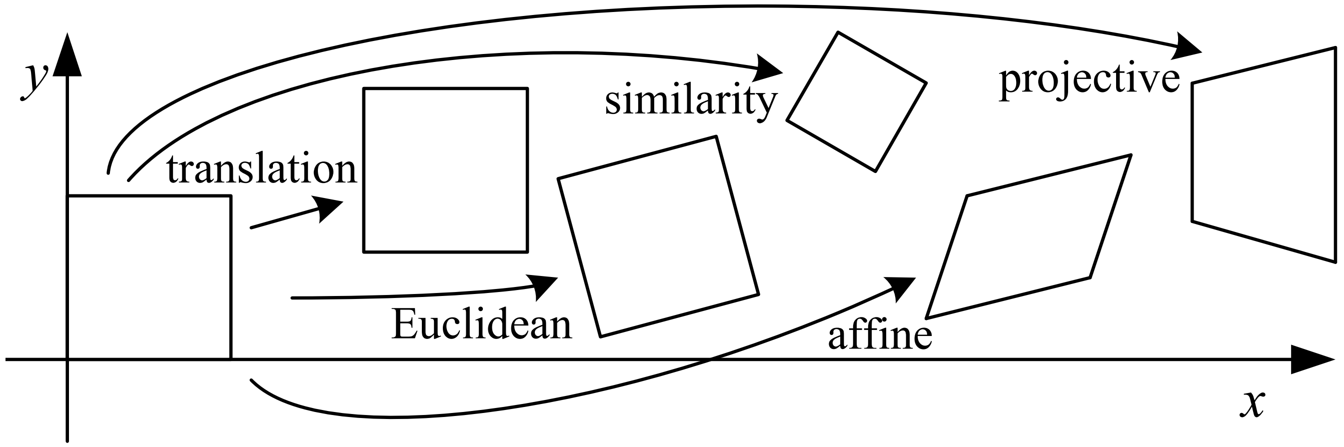

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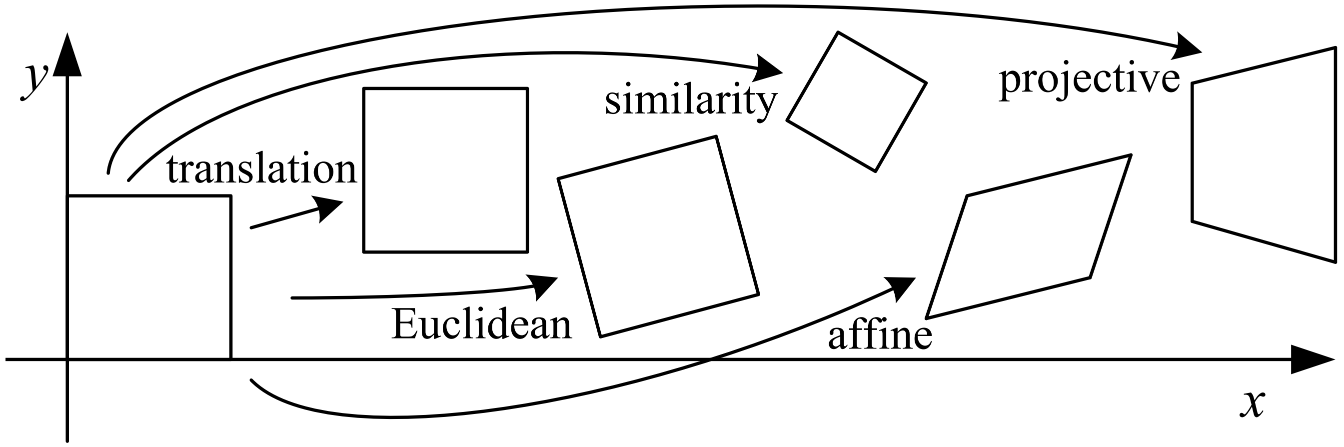
$$\tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{H}}\tilde{\boldsymbol{x}}$$

where $\tilde{\boldsymbol{H}}$ is an arbitrary 3×3 matrix.

$\tilde{\boldsymbol{H}}$ is homogenous:

Two $\tilde{\boldsymbol{H}}$ matrices that differ only by a scale factor are equivalent.



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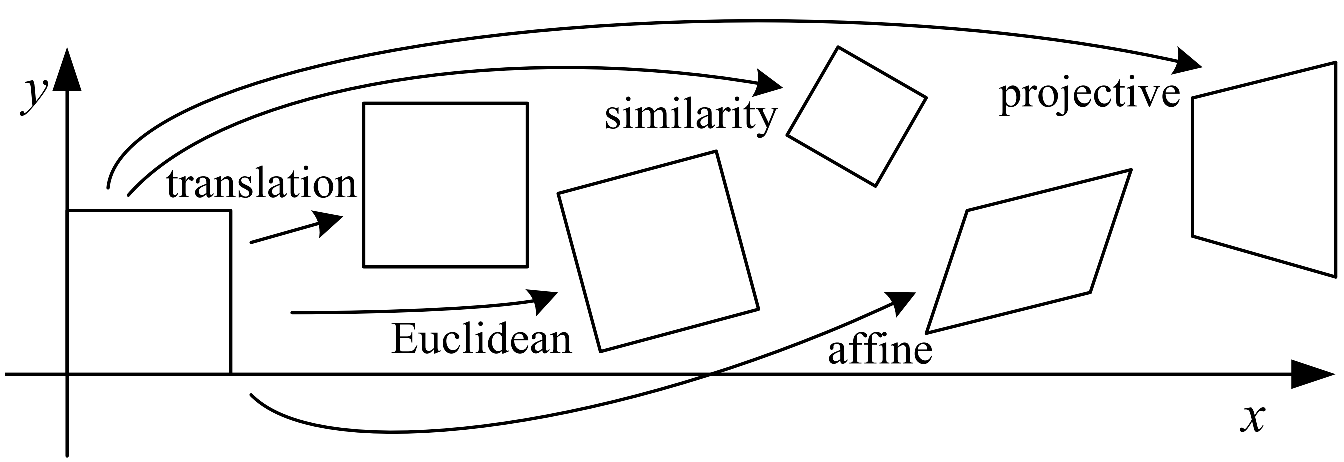

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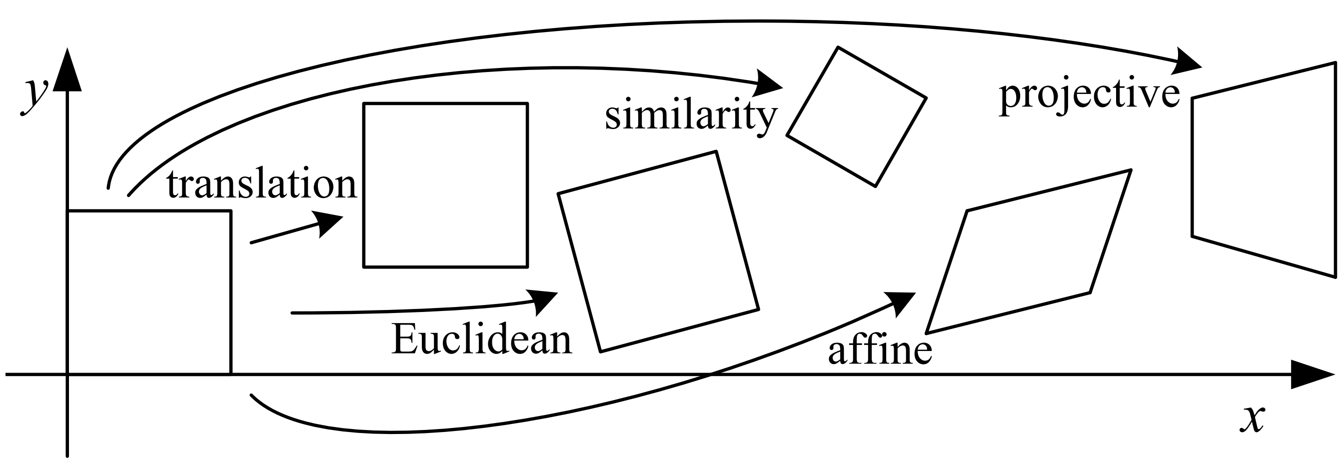
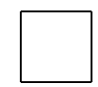
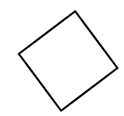
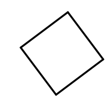
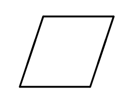
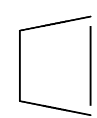
Preserves straight lines

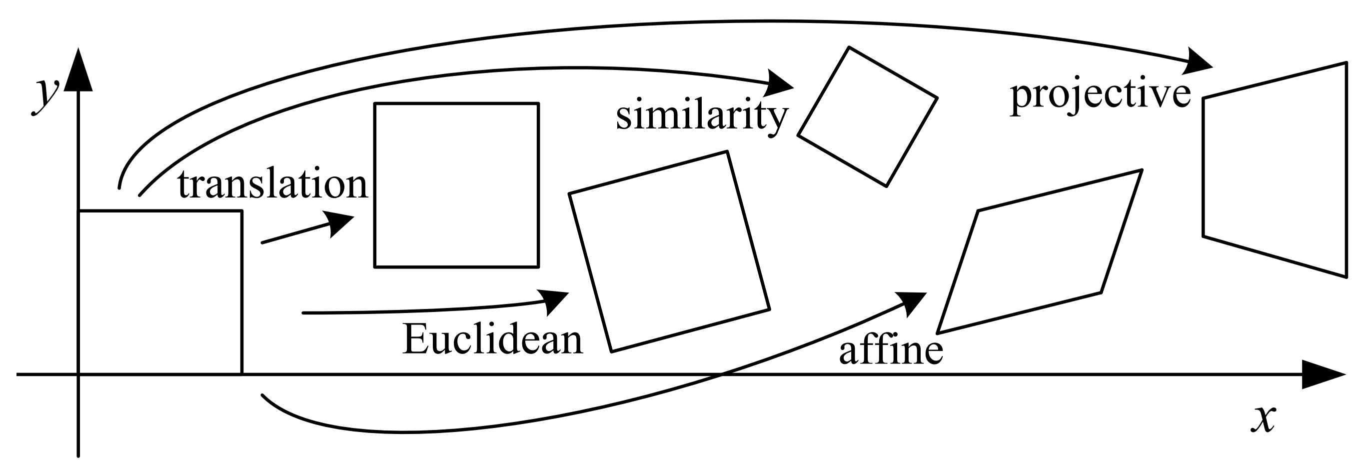


Summary of 2D Transformations

Nested set of groups

- ❖ Closed under composition
- ❖ Each transformation has an inverse that is a member of the same group

Transformation	**Matrix**	**# DoF**	**Preserves**	**Icon**
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	



Co-vectors

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We now know how to transform points.
How do we transform lines?

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Co-vectors

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How do we transform lines?

$$\tilde{\boldsymbol{l}} \cdot \tilde{\boldsymbol{x}} = 0$$

$$\tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{H}} \tilde{\boldsymbol{x}}$$

$$\tilde{\boldsymbol{l}}' \cdot \tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{l}}'^T \tilde{\boldsymbol{H}} \tilde{\boldsymbol{x}} = (\tilde{\boldsymbol{H}}^T \tilde{\boldsymbol{l}}')^T \tilde{\boldsymbol{x}} = \tilde{\boldsymbol{l}} \cdot \tilde{\boldsymbol{x}} = 0$$

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Thus

Co-vectors

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How do we transform lines?

$$\tilde{\boldsymbol{l}} \cdot \tilde{\boldsymbol{x}} = 0$$

$$\tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{H}} \tilde{\boldsymbol{x}}$$

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Thus

$$\tilde{\boldsymbol{l}}' = \tilde{\boldsymbol{H}}^{-T} \tilde{\boldsymbol{l}}.$$

Co-vectors

We now know how to transform points.
How do we transform lines?

$$\tilde{\boldsymbol{l}} \cdot \tilde{\boldsymbol{x}} = 0$$

$$\tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{H}}\tilde{\boldsymbol{x}}$$

$$\tilde{\boldsymbol{l}}' \cdot \tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{l}}'^T \tilde{\boldsymbol{H}}\tilde{\boldsymbol{x}} = (\tilde{\boldsymbol{H}}^T \tilde{\boldsymbol{l}}')^T \tilde{\boldsymbol{x}} = \tilde{\boldsymbol{l}} \cdot \tilde{\boldsymbol{x}} = 0$$

Thus

$$\tilde{\boldsymbol{l}}' = \tilde{\boldsymbol{H}}^{-T} \tilde{\boldsymbol{l}}.$$

i.e., the action of a projective transformation on a *co-vector* such as a 2D line can be represented by the transposed inverse of the matrix.

Outline

- Geometric primitives
- 2D transformations
- **3D transformations**
- 3D rotations
- 3D to 2D projections
- Lens Distortions

3D Translation

3D translations can be written as $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{t}$ or

$$\boldsymbol{x}' = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \end{bmatrix} \bar{\boldsymbol{x}}$$

where $\boldsymbol{I}$ is the (3×3) identity matrix

Euclidean Transformation (3D Rotation + Translation)

$$\boldsymbol{x}' = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$$

$$\boldsymbol{x}' = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \end{bmatrix} \bar{\boldsymbol{x}}$$

where $\boldsymbol{R}$ is a 3×3 orthonormal rotation matrix with $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$ and $|\boldsymbol{R}| = 1$

Preserves Euclidean distances

Similarity Transformation

$$\boldsymbol{x}' = s\boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$$

$$\boldsymbol{x}' = \begin{bmatrix} s\boldsymbol{R} & \boldsymbol{t} \end{bmatrix} \bar{\boldsymbol{x}}$$

Preserves angles

Affine Transformation

$\boldsymbol{x}' = \boldsymbol{A}\bar{\boldsymbol{x}}$, where $\boldsymbol{A}$ is an arbitrary 3×4 matrix

$$\boldsymbol{x}' = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \end{bmatrix} \bar{\boldsymbol{x}}$$

Preserves parallelism

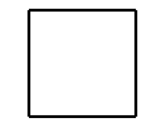
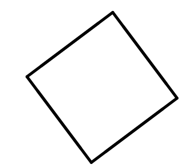
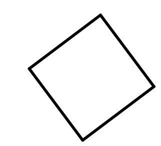
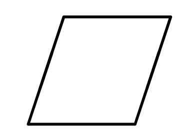
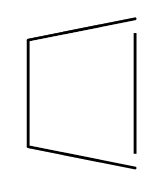
Projective Transformation (Homography)

$$\tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{H}}\tilde{\boldsymbol{x}}$$

where $\tilde{\boldsymbol{H}}$ is an arbitrary 4×4 homogeneous matrix

Preserves straight lines

Summary of 3D Transformations

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{3\times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{3\times 4}$	6	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{3\times 4}$	7	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{3\times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{4\times 4}$	15	straight lines	

Outline

- Geometric primitives
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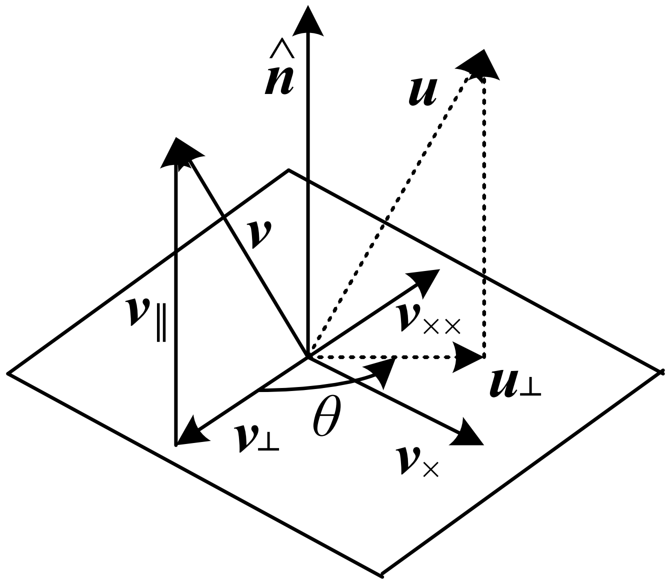
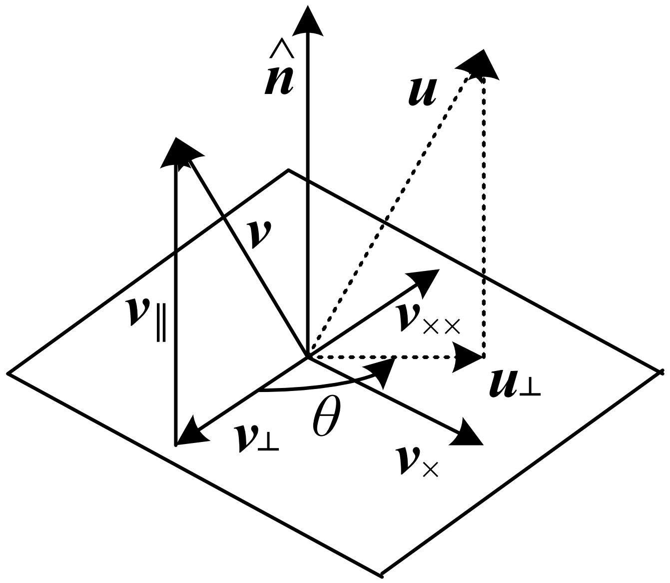
3D Rotations: Axis/Angle Representation



3D Rotations: Axis/Angle Representation

$$\boldsymbol{\omega} = \theta \hat{\boldsymbol{n}}$$



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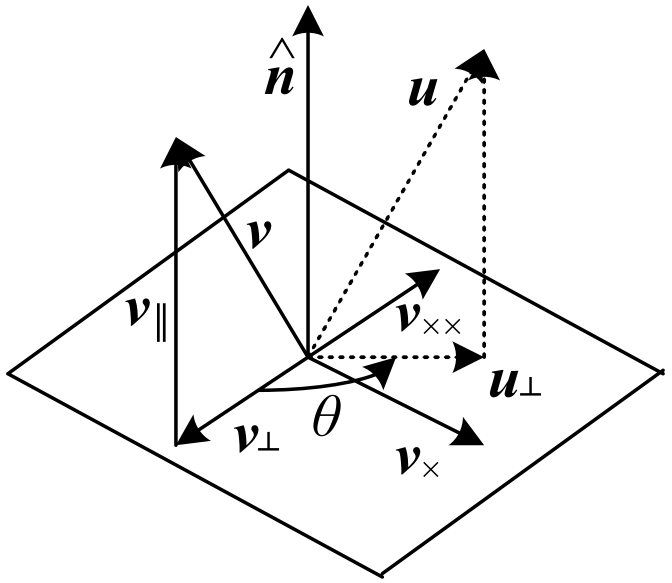
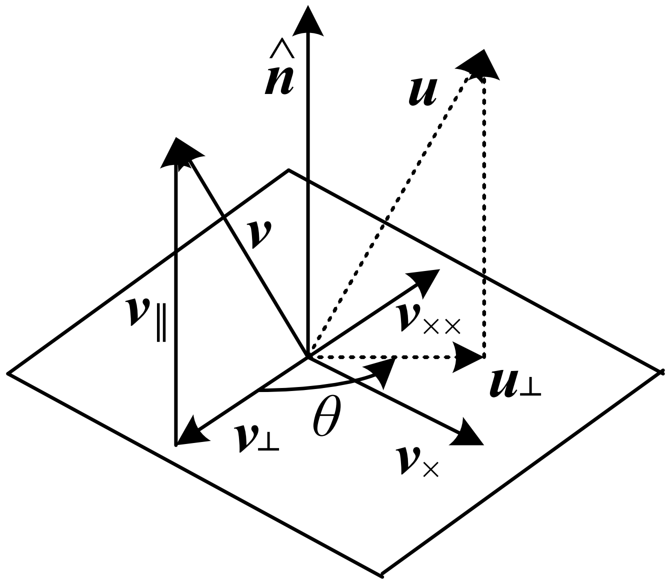


3D Rotations: Axis/Angle Representation

$$\boldsymbol{\omega} = \theta \hat{\boldsymbol{n}}$$

Amount of rotation



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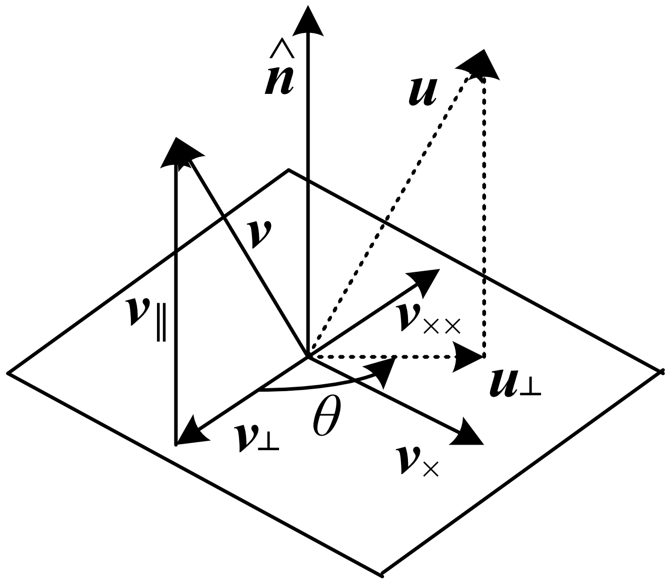

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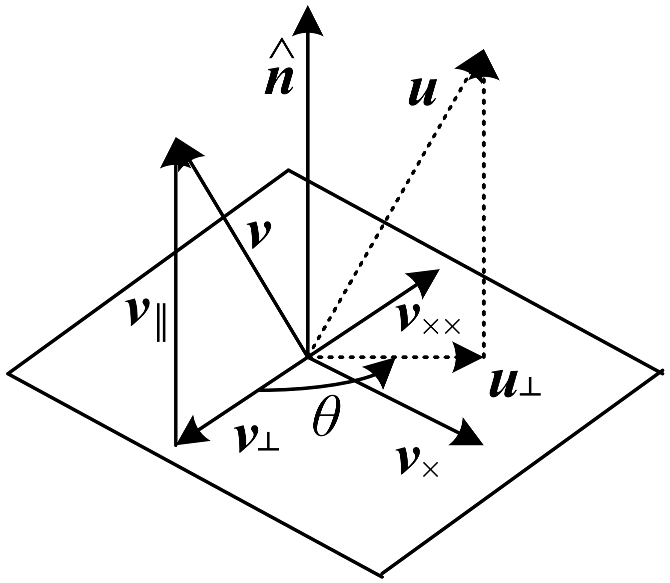
$$\boldsymbol{\omega} = \theta \hat{\boldsymbol{n}}$$

Amount of rotation

Axis of rotation



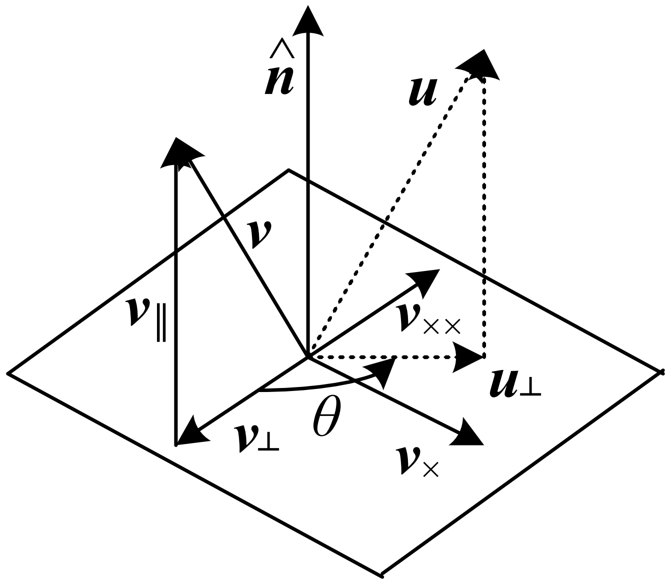
3D Rotations: Axis/Angle Representation

$$\boldsymbol{\omega} = \theta \hat{\boldsymbol{n}}$$

Amount of rotation

Axis of rotation

Let $\boldsymbol{u}$ be the result of rotating vector $\boldsymbol{v}$ about axis $\hat{\boldsymbol{n}}$ by the angle θ.



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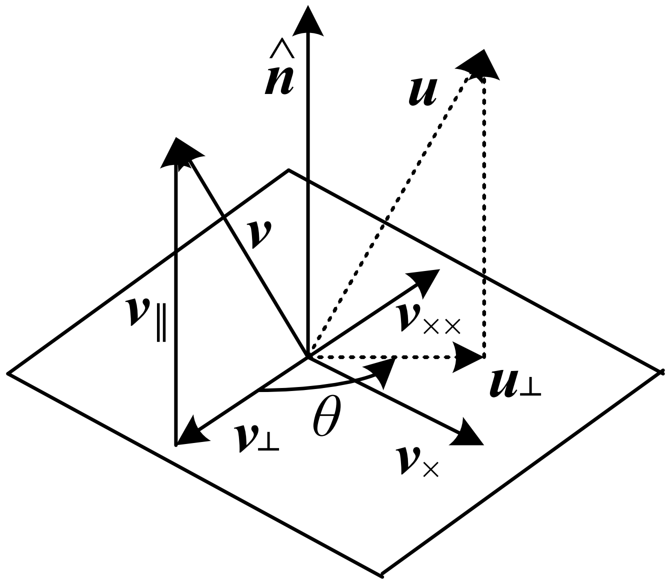
$$\boldsymbol{\omega} = \theta \hat{\boldsymbol{n}}$$

Amount of rotation   Axis of rotation

Let $\boldsymbol{u}$ be the result of rotating vector $\boldsymbol{v}$ about axis $\hat{\boldsymbol{n}}$ by the angle θ.

First, project the vector $\boldsymbol{v}$ onto the axis $\hat{\boldsymbol{n}}$:



3D Rotations: Axis/Angle Representation

$$\boldsymbol{\omega} = \theta\hat{\boldsymbol{n}}$$

Amount of rotation   Axis of rotation

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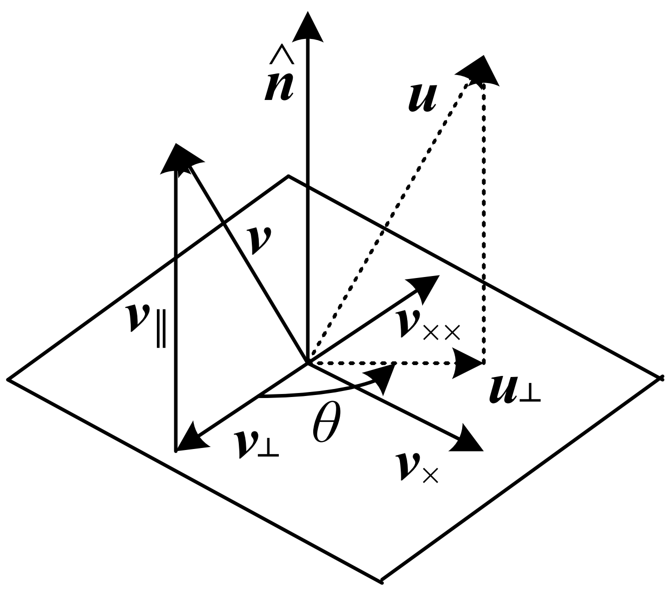

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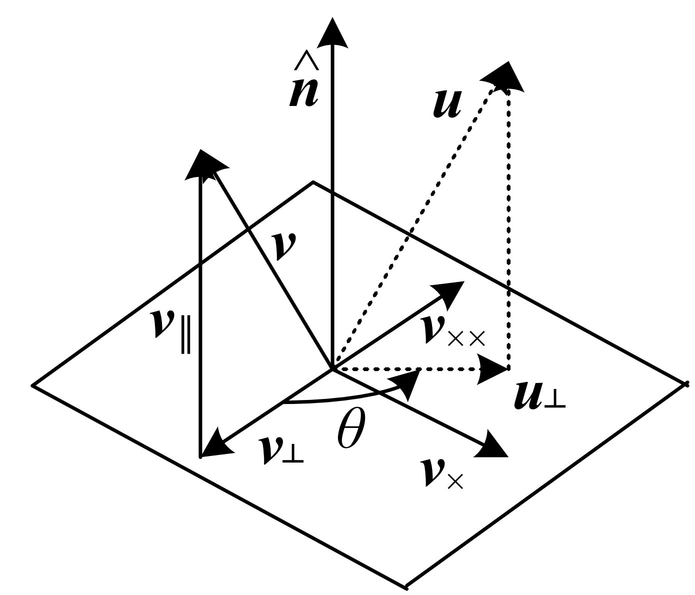

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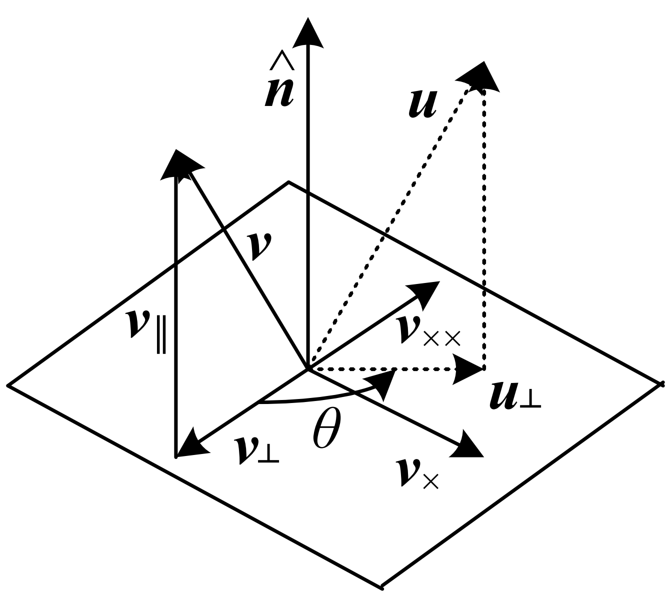

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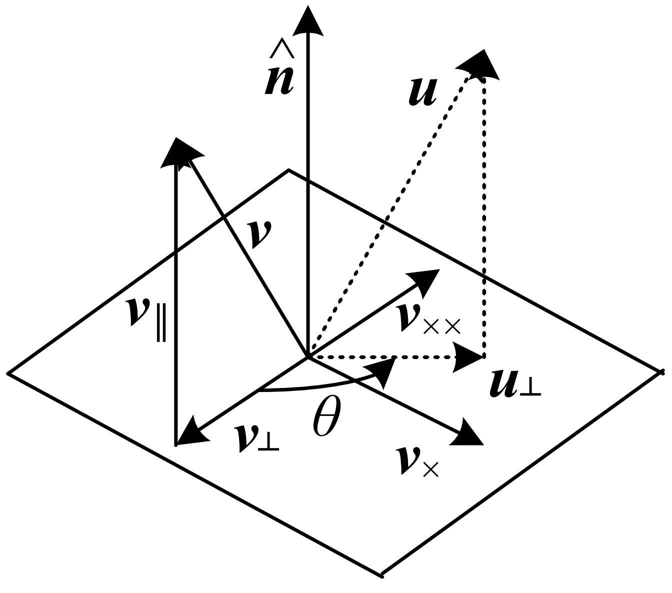

3D Rotations: Axis/Angle Representation

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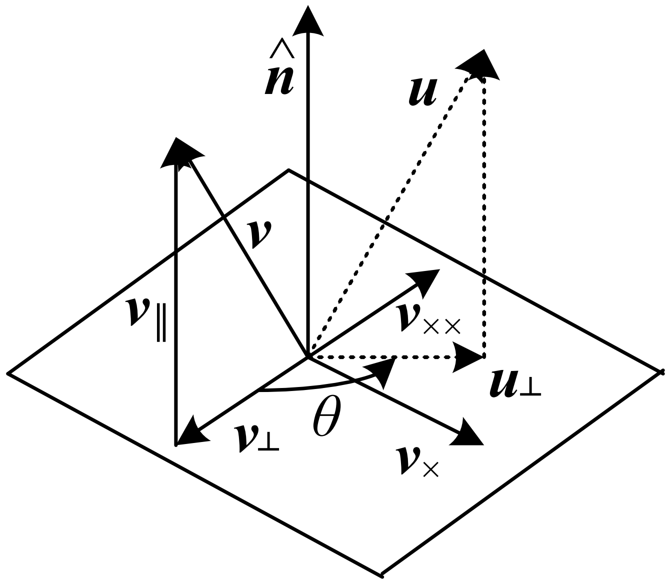
$$\boldsymbol{v}_\times = \hat{\boldsymbol{n}} \times \boldsymbol{v} = [\hat{\boldsymbol{n}}]_\times \boldsymbol{v},$$



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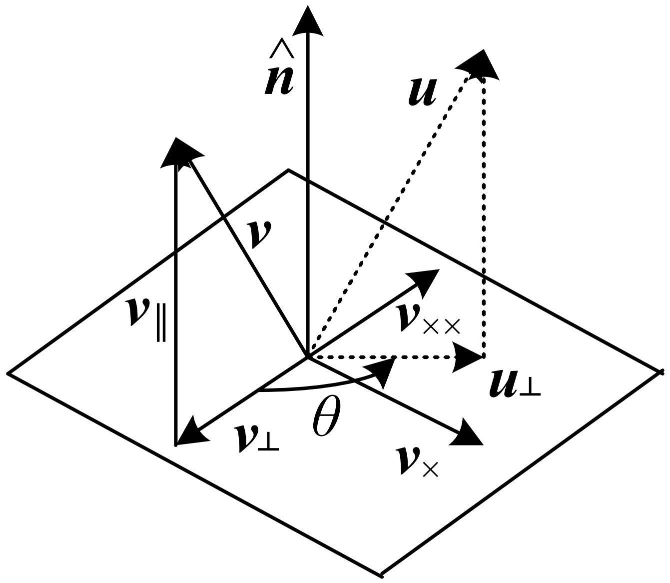

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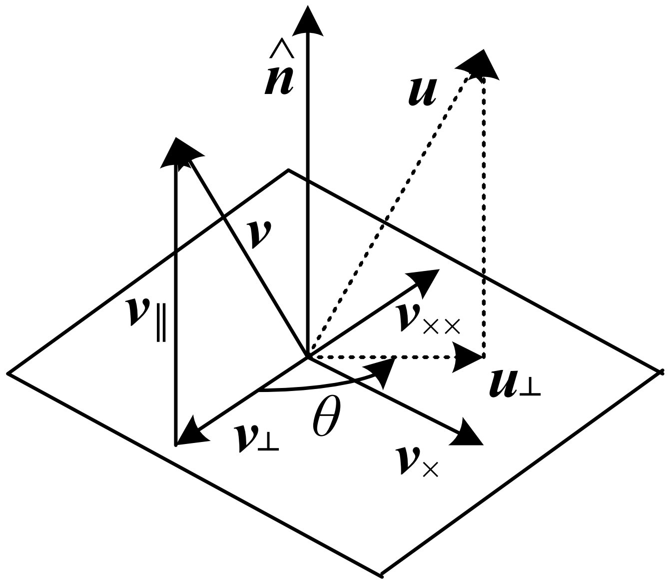

3D Rotations: Axis/Angle Representation

We can rotate this vector by $90°$ using the cross product,

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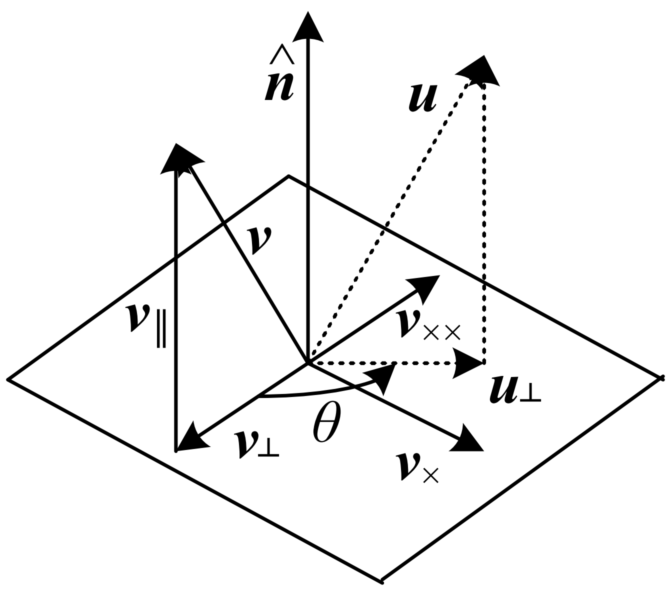
Note that rotating this vector by another $90°$ is equivalent to taking the cross product again,

$$\boldsymbol{v}_{\times\times} = \hat{\boldsymbol{n}} \times \boldsymbol{v}_\times = [\hat{\boldsymbol{n}}]^2_\times \boldsymbol{v} = -\boldsymbol{v}_\perp,$$

and hence

$$\boldsymbol{v}_\parallel = \boldsymbol{v} - \boldsymbol{v}_\perp = \boldsymbol{v} + \boldsymbol{v}_{\times\times} = (\boldsymbol{I} + [\hat{\boldsymbol{n}}]^2_\times)\boldsymbol{v}.$$



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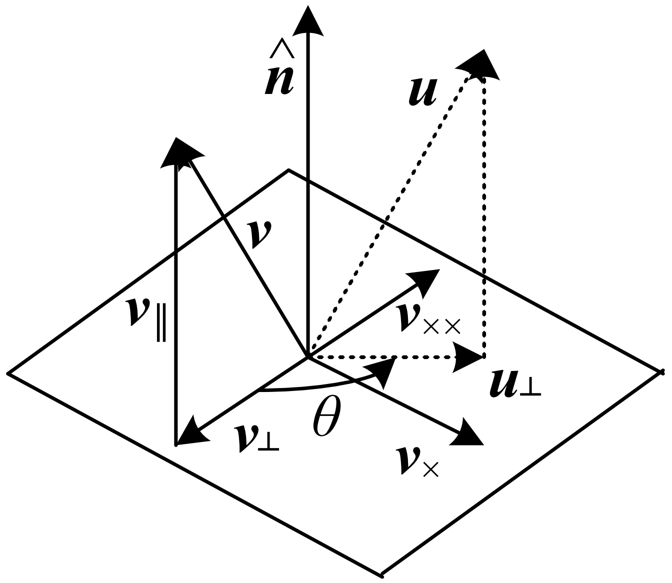


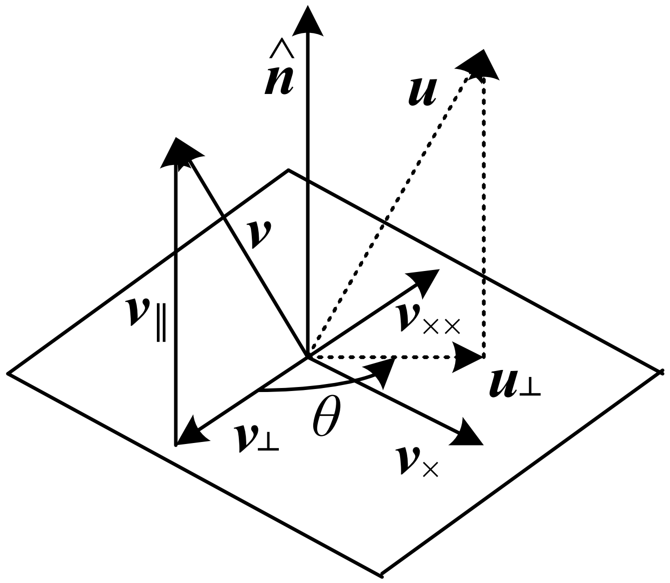
3D Rotations: Axis/Angle Representation

We can now compute the in-plane component of the rotated vector $\boldsymbol{u}$ as

$$\boldsymbol{u}_\perp = \cos\theta \boldsymbol{v}_\perp + \sin\theta \boldsymbol{v}_\times = (\sin\theta[\hat{\boldsymbol{n}}]_\times - \cos\theta[\hat{\boldsymbol{n}}]_\times^2)\boldsymbol{v}.$$

Putting all these terms together, we obtain the final rotated vector as

$$\boldsymbol{u} = \boldsymbol{u}_\perp + \boldsymbol{v}_\parallel = (\boldsymbol{I} + \sin\theta[\hat{\boldsymbol{n}}]_\times + (1-\cos\theta)[\hat{\boldsymbol{n}}]_\times^2)\boldsymbol{v}.$$



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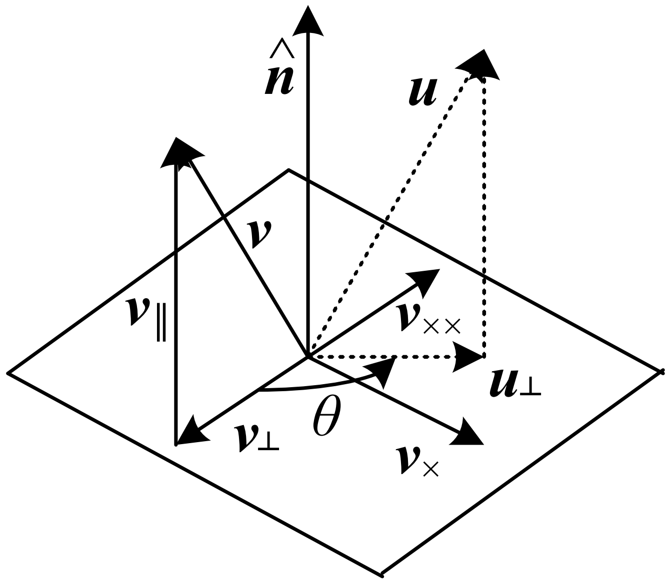
We can therefore write the rotation matrix corresponding to a rotation by θ around an axis $\hat{\boldsymbol{n}}$ as



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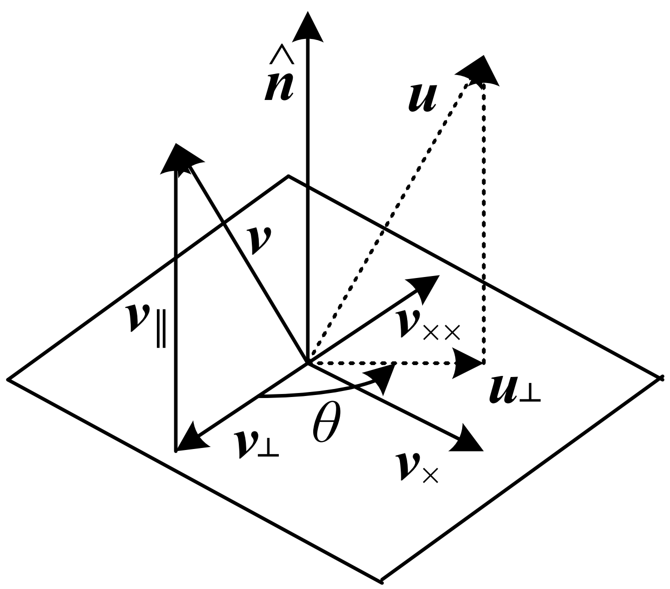

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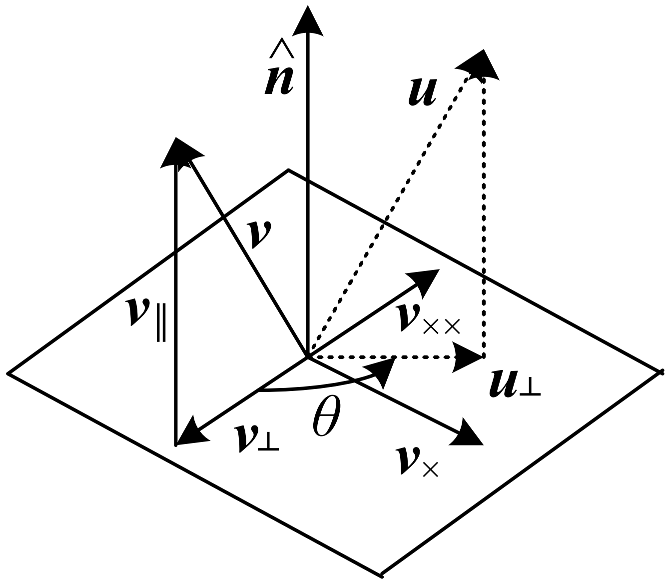
$$\boldsymbol{R}(\hat{\boldsymbol{n}}, \theta) = \boldsymbol{I} + \sin\theta[\hat{\boldsymbol{n}}]_\times + (1 - \cos\theta)[\hat{\boldsymbol{n}}]^2_\times$$ **(Rodriquez' formula)**



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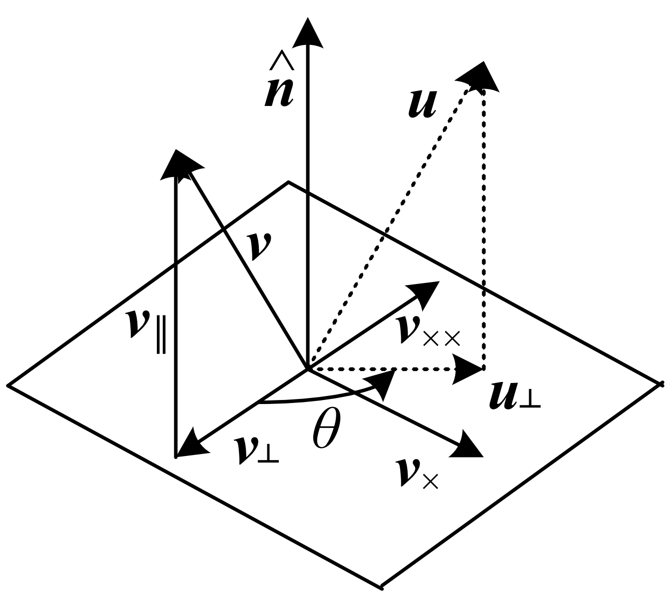
For small rotations:

$$\boldsymbol{R}(\boldsymbol{\omega}) \approx \boldsymbol{I} + \sin\theta [\hat{\boldsymbol{n}}]_\times \approx \boldsymbol{I} + [\theta \hat{\boldsymbol{n}}]_\times = \begin{bmatrix} 1 & -\omega_z & \omega_y \\ \omega_z & 1 & -\omega_x \\ -\omega_y & \omega_x & 1 \end{bmatrix}$$



Unit Quaternions

$$\boldsymbol{q} = (x, y, z, w) \text{ with } \|\boldsymbol{q}\| = 1$$

q and ***-q*** represent the same rotation.

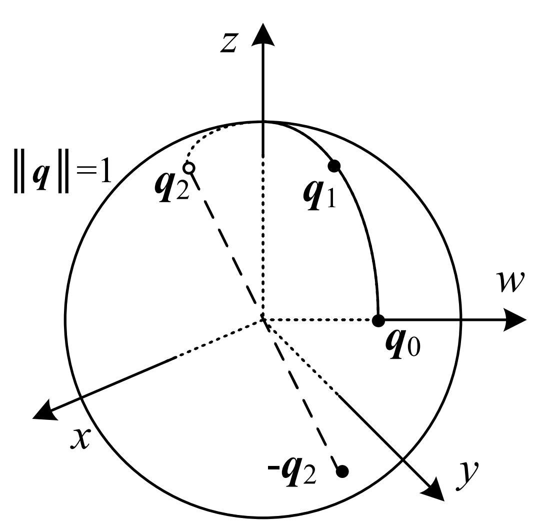
Quaternions can be derived from the axis/angle representation through the formula

$$\boldsymbol{q} = (\boldsymbol{v}, w) = (\sin\frac{\theta}{2}\hat{\boldsymbol{n}}, \cos\frac{\theta}{2}),$$

where $\hat{\boldsymbol{n}}$ and θ are the rotation axis and angle.

Rodriguez' formula now becomes (see textbook):

$$\begin{aligned} \boldsymbol{R}(\hat{\boldsymbol{n}}, \theta) &= \boldsymbol{I} + \sin\theta[\hat{\boldsymbol{n}}]_\times + (1 - \cos\theta)[\hat{\boldsymbol{n}}]_\times^2 \\ &= \boldsymbol{I} + 2w[\boldsymbol{v}]_\times + 2[\boldsymbol{v}]_\times^2. \end{aligned}$$



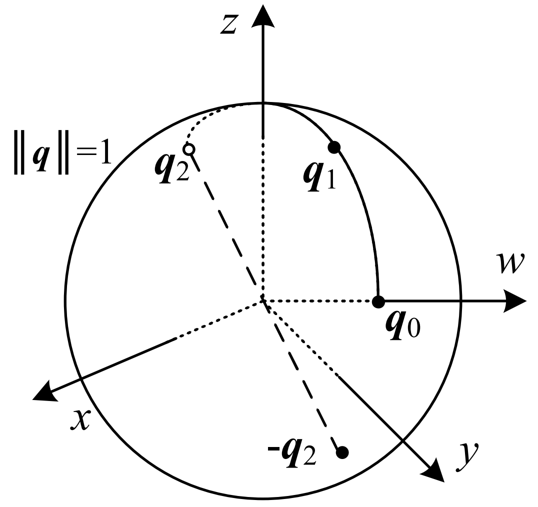
Quaternion Algebra

$$\boldsymbol{q} = (\boldsymbol{v}, w) = (\sin\frac{\theta}{2}\hat{\boldsymbol{n}}, \cos\frac{\theta}{2})$$

Composition (multiplication): $\boldsymbol{q}_2 = \boldsymbol{q}_0\boldsymbol{q}_1 = (\boldsymbol{v}_0 \times \boldsymbol{v}_1 + w_0\boldsymbol{v}_1 + w_1\boldsymbol{v}_0,\ w_0w_1 - \boldsymbol{v}_0 \cdot \boldsymbol{v}_1)$

$$\boldsymbol{R}(\boldsymbol{q}_2) = \boldsymbol{R}(\boldsymbol{q}_0)\boldsymbol{R}(\boldsymbol{q}_1).$$

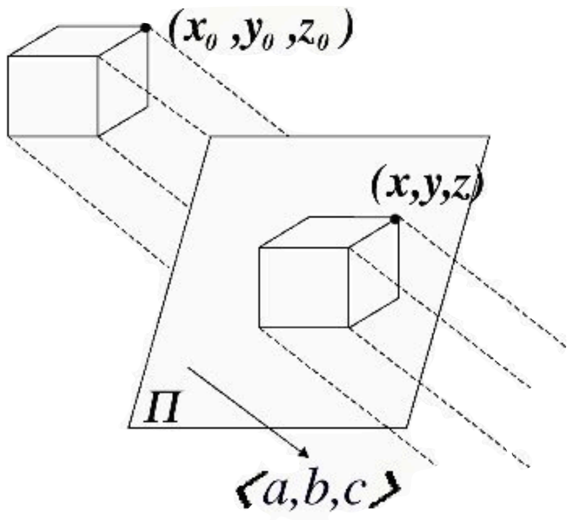
Inverse: flip the sign of *v* or *w* (but not both).
i.e., if $\boldsymbol{q} = (\boldsymbol{v}, w)$, then $\boldsymbol{q}^{-1} = (-\boldsymbol{v}, w) = (\boldsymbol{v}, -w)$.



Outline

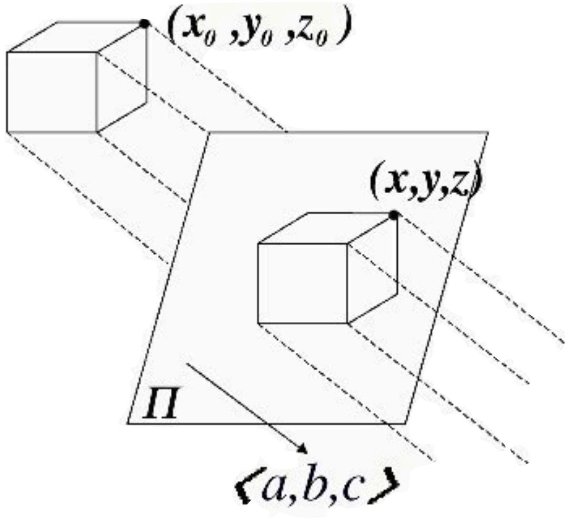
- Geometric primitives
- 2D transformations
- 3D transformations
- 3D rotations
- **3D to 2D projections**
- Lens Distortions

Orthographic (Parallel) Projection



Orthographic (Parallel) Projection

- Reasonable approximation to perspective projection when % depth variation within field of view is small.



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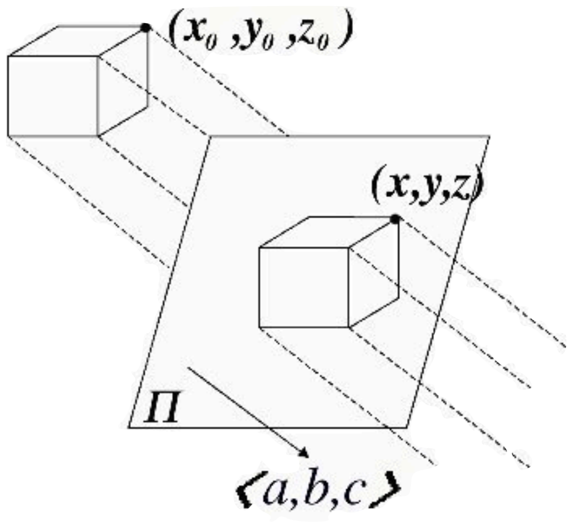

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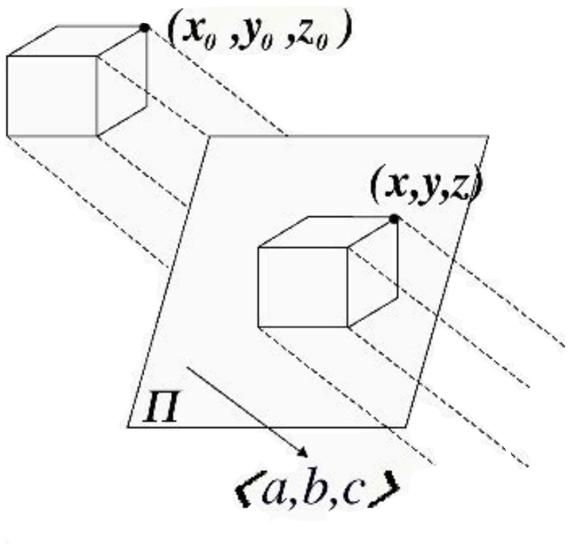

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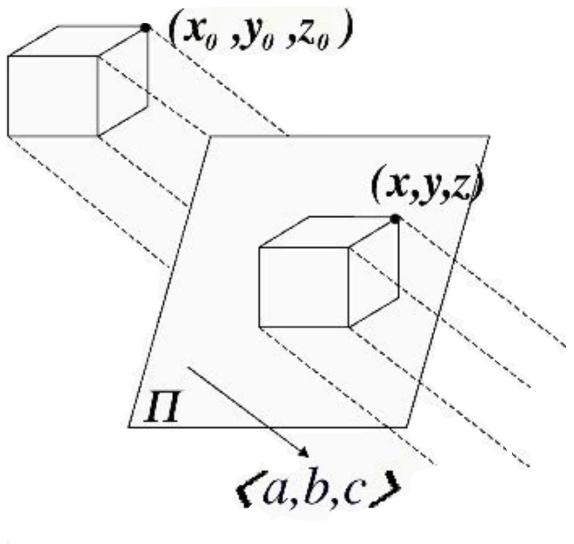

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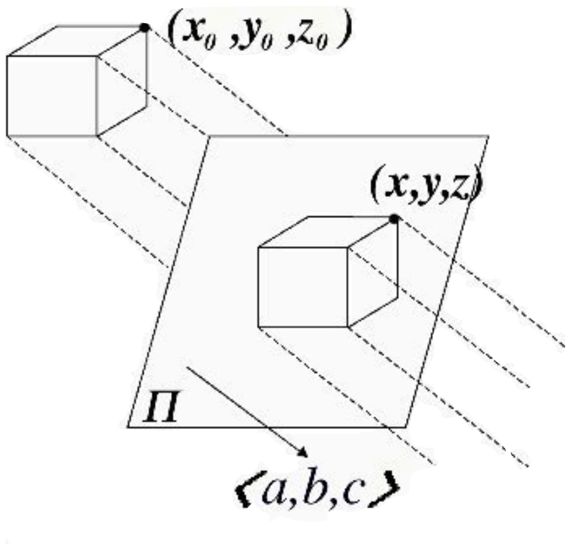

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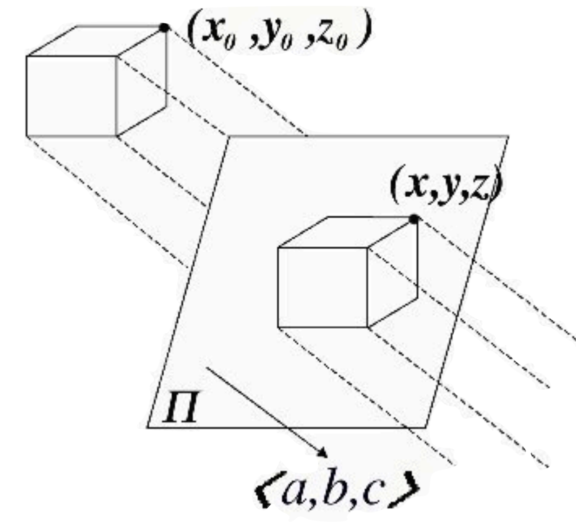

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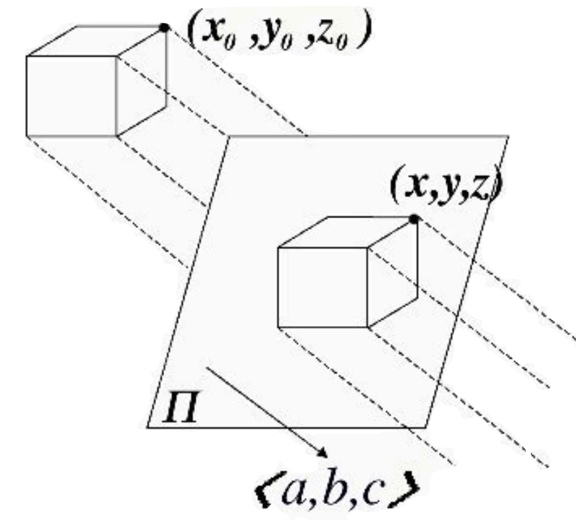

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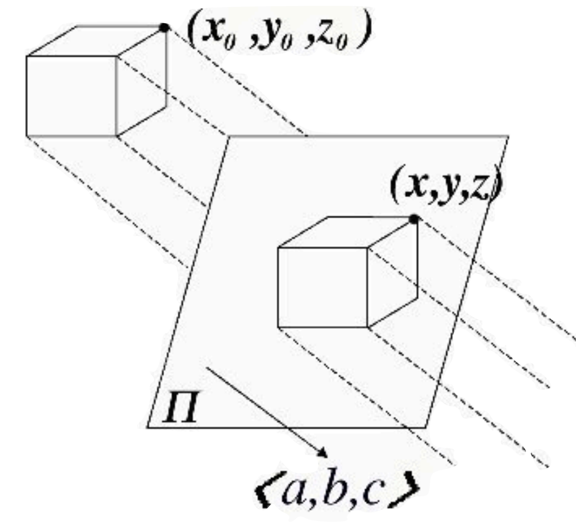

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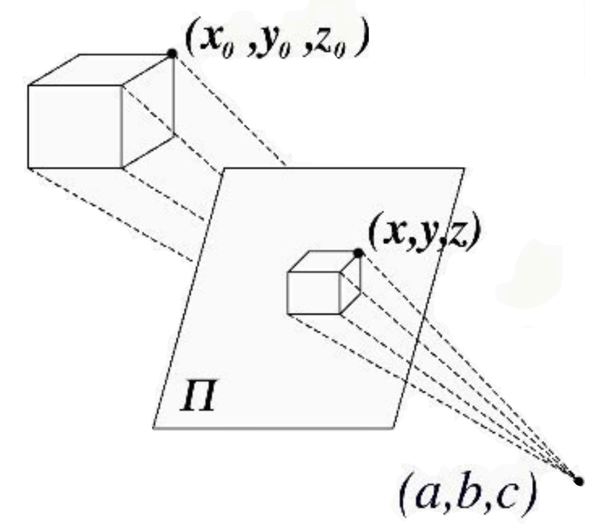
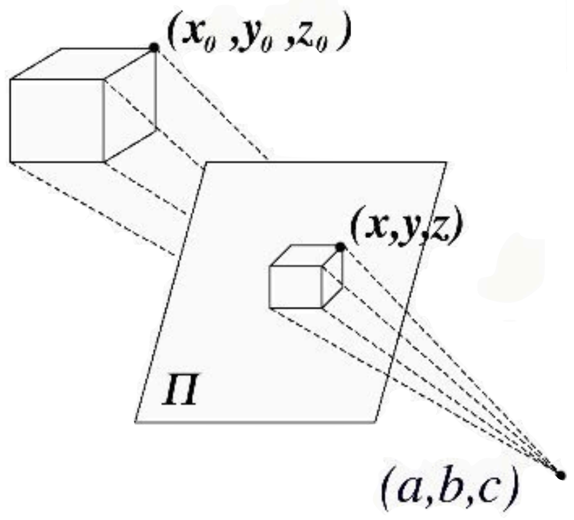
Perspective Projection



Perspective Projection

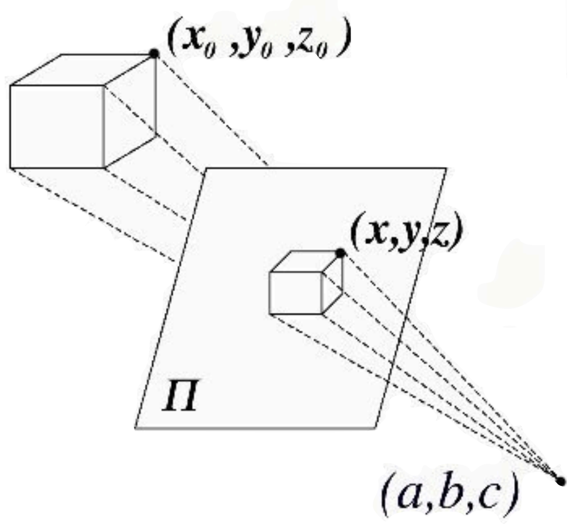
- Points projected onto image plane by dividing them by their z component.



Perspective Projection

❖ Points projected onto image plane by dividing them by their z component.

$$\bar{\boldsymbol{x}} = \mathcal{P}_z(\boldsymbol{p}) = \begin{bmatrix} x/z \\ y/z \\ 1 \end{bmatrix}$$



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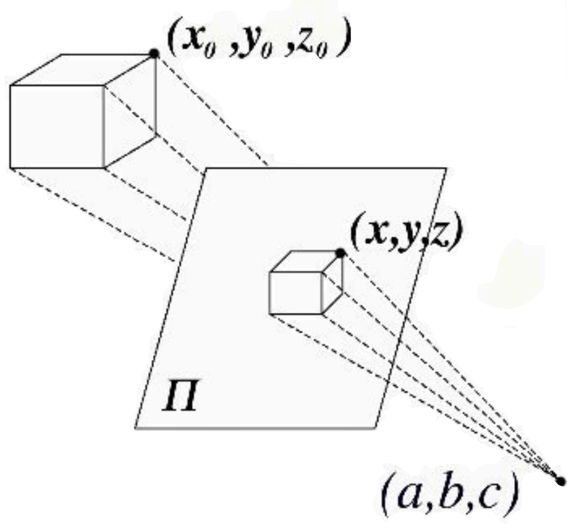

- In homogeneous coordinates:

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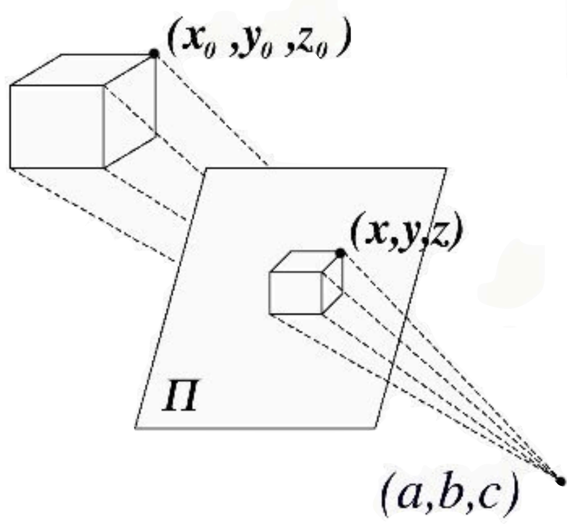

- In homogeneous coordinates:

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tilde{\boldsymbol{p}},$$

Camera Intrinsics

$$\tilde{\boldsymbol{x}}_s = \boldsymbol{K} \left[\begin{array}{c|c} \boldsymbol{R} & \boldsymbol{t} \end{array} \right] \bar{\boldsymbol{p}}_w = \boldsymbol{P} \bar{\boldsymbol{p}}_w$$

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2D image projection

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3D world point

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2D image projection

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Extrinsic (rotation + translation) matrix

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$$\boldsymbol{K} = \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\boldsymbol{K} = \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

f_x and f_y: encode focal length and pixel spacing, which may be slightly different in x and y dimensions.

Camera Intrinsics

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2D image projection

Intrinsic (calibration) matrix

Extrinsic (rotation + translation) matrix

Projection matrix

3D world point

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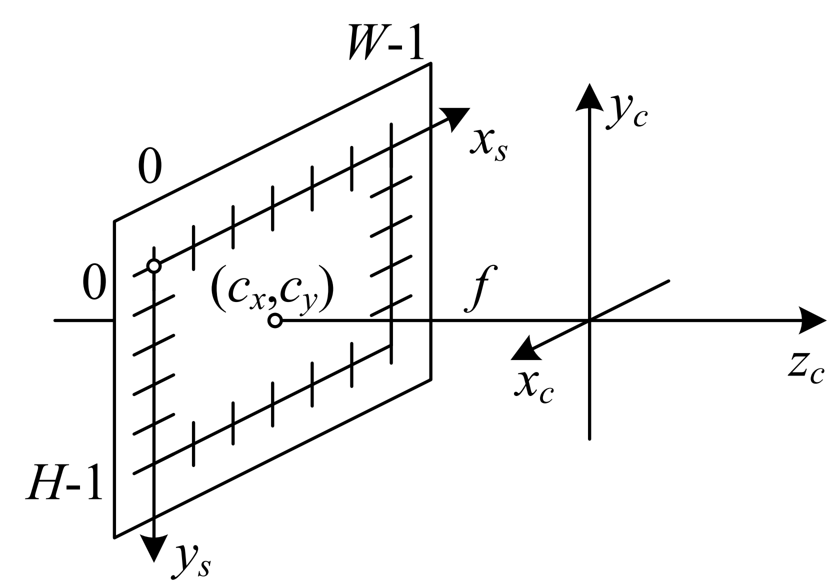

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❖ Focal length can be measured either in pixels or in mm.

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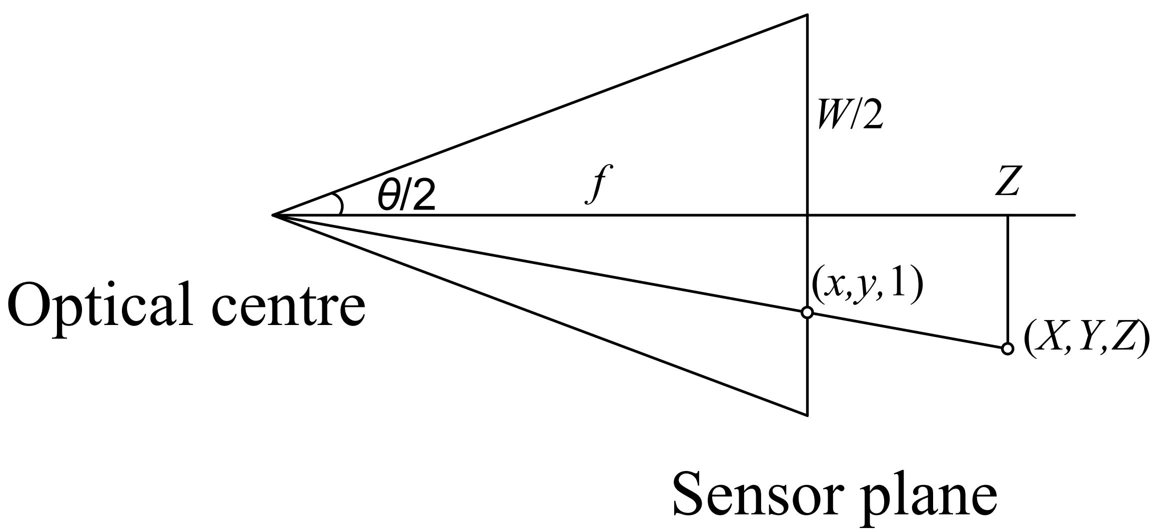

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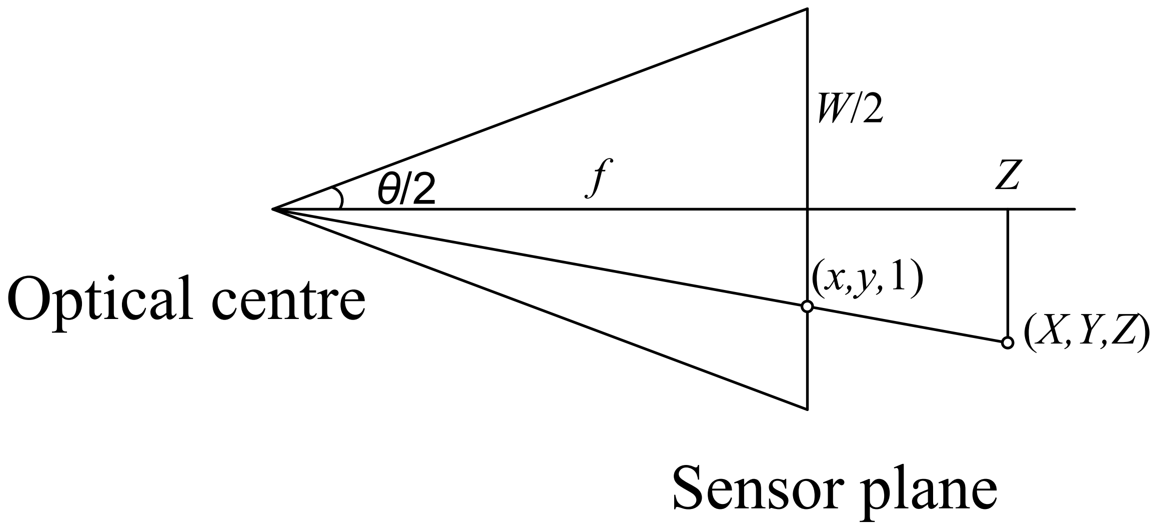

- Example: Consider the FLIR BlackFly S BFS-PGE-122S6C-C paired with a 10mm lens:
  - Resolution: 4096 x 3000 pixels
  - Sensor width: 1.1” = 27.94mm

Outline

- Geometric primitives
- 2D transformations
- 3D transformations
- 3D rotations
- 3D to 2D projections
- **Lens Distortions**

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