



Recursion

Chapter 5

Outline

- Induction
- Linear recursion
 - Example 1: Factorials
 - Example 2: Powers
 - Example 3: Reversing an array
- Binary recursion
 - Example 1: The Fibonacci sequence
 - Example 2: The Tower of Hanoi
- Drawbacks and pitfalls of recursion

Outcomes

- By understanding this lecture you should be able to:
 - Use induction to prove the correctness of a recursive algorithm.
 - Identify the base case for an inductive solution
 - Design and analyze linear and binary recursion algorithms
 - Identify the overhead costs of recursion
 - Avoid errors commonly made in writing recursive algorithms

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Divide and Conquer

- When faced with a difficult problem, a classic technique is to break it down into smaller parts that can be solved more easily.
- Recursion uses induction to do this.



History of Induction

- Implicit use of induction goes back at least to Euclid's proof that the number of primes is infinite (c. 300 BC).
- The first explicit formulation of the principle is due to Pascal (1665).



Euclid of Alexandria,
"The Father of Geometry"
c. 300 BC



Blaise Pascal, 1623 - 1662

Induction: Review

- Induction is a mathematical method for proving that a statement is true for a (possibly infinite) sequence of objects.
- There are two things that must be proved:
 1. **The Base Case:** The statement is true for the first object
 2. **The Inductive Step:** If the statement is true for a given object, it is also true for the next object.
- If these two statements hold, then the statement holds for all objects.

Induction Example: An Arithmetic Sum

- Claim: $\sum_{i=0}^n i = \frac{1}{2}n(n+1) \quad \forall n \in \mathbb{N}$
- Proof:

1. Base Case. The statement holds for $n = 0$:

$$\sum_{i=0}^n i = \sum_{i=0}^0 i = 0$$

$$\frac{1}{2}n(n+1) = \frac{1}{2}0(0+1) = 0$$



1. Inductive Step. If the claim holds for $n = k$, then it also holds for $n = k+1$.

$$\sum_{i=0}^{k+1} i = k+1 + \sum_{i=0}^k i = k+1 + \frac{1}{2}k(k+1) = \frac{1}{2}(k+1)(k+2)$$



Recursive Divide and Conquer

- You are given a problem input that is too big to solve directly.
- You imagine,
 - “Suppose I had a friend who could give me the answer to the same problem with slightly smaller input.”
 - “Then I could easily solve the larger problem.”
- In recursion this “friend” will actually be another instance (clone) of yourself.



Tai (left) and Snuppy (right): the first puppy clone.

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Recall: Design Pattern

- A template for a software solution that can be applied to a variety of situations.
- Main elements of solution are described in the abstract.
- Can be specialized to meet specific circumstances.

Linear Recursion Design Pattern

- **Test for base cases**

- Begin by testing for a set of base cases (there should be at least one).
- Every possible chain of recursive calls **must** eventually reach a base case, and the handling of each base case should not use recursion.

- ***Recurse once***

- Perform a single recursive call. (This recursive step may involve a test that decides which of several possible recursive calls to make, but it should ultimately choose to make just one of these calls each time we perform this step.)
- Define each possible recursive call so that it makes **progress** towards a base case.

Example 1

- The factorial function:
 - $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$

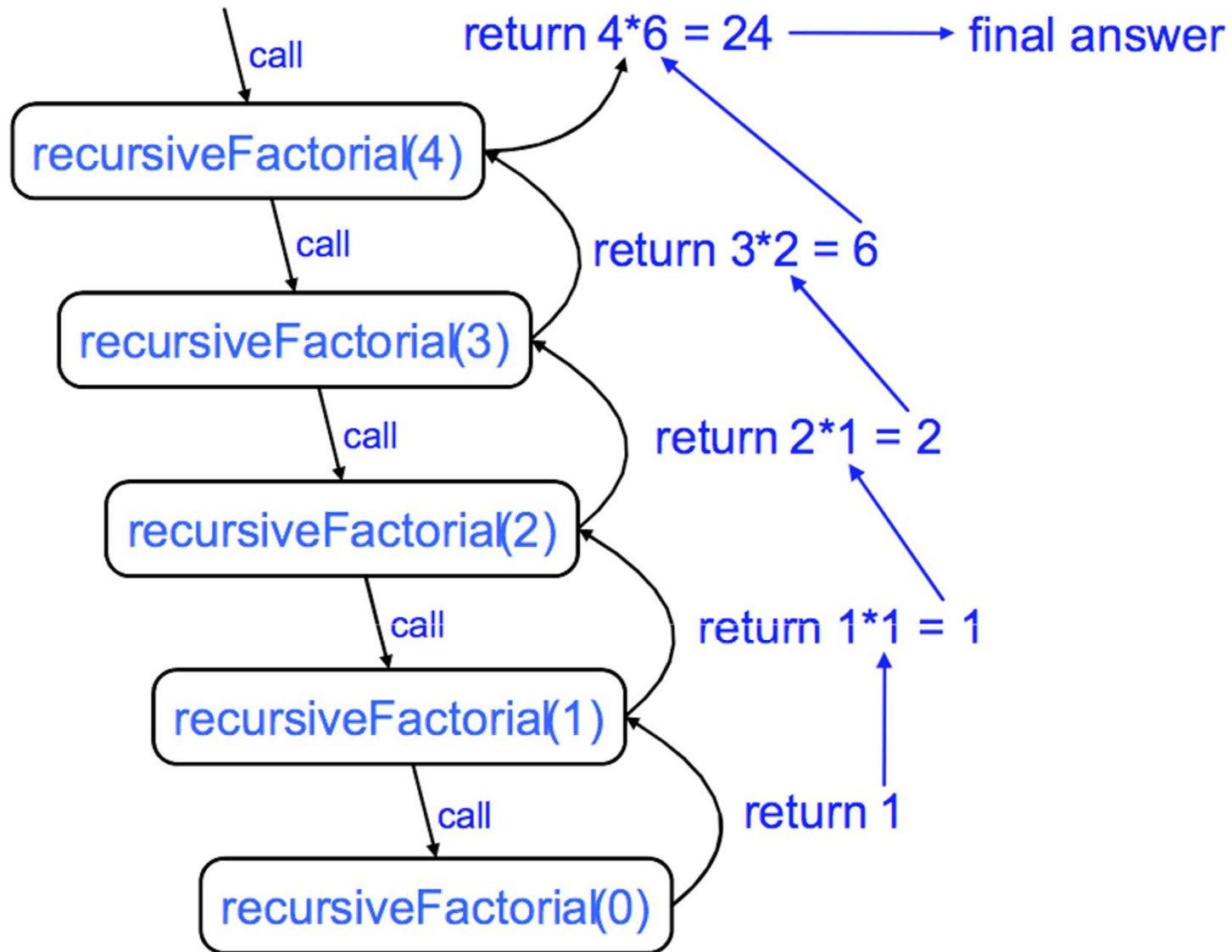
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & \text{else} \end{cases}$$

- As a Java method:

```
// recursive factorial function
public static int recursiveFactorial(int n) {
    if (n == 0) return 1; // base case
    else return n * recursiveFactorial(n- 1); // recursive case
}
```

Tracing Recursion



Linear Recursion

- recursiveFactorial is an example of **linear** recursion: only one recursive call is made per stack frame.
- Since there are n recursive calls, this algorithm has $O(n)$ run time.

```
// recursive factorial function
public static int recursiveFactorial(int n) {
    if (n == 0) return 1;      // base case
    else return n * recursiveFactorial(n- 1);    // recursive case
}
```

Example 2: Computing Powers

- The power function, $p(x,n) = x^n$, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x,n-1) & \text{otherwise} \end{cases}$$

- Assume multiplication takes constant time (independent of value of arguments).
- This leads to a power function that runs in $O(n)$ time (for we make n recursive calls).
- **Can we do better than this?**

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, (n-1)/2)^2 & \text{if } n > 0 \text{ is odd} \\ p(x, n/2)^2 & \text{if } n > 0 \text{ is even} \end{cases}$$

- For example,

$$2^4 = 2^{(4/2)^2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$$

$$2^5 = 2^{1+(4/2)^2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$$

$$2^6 = 2^{(6/2)^2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$$

$$2^7 = 2^{1+(6/2)^2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$$

A Recursive Squaring Method

Algorithm Power(x , n):

Input: A number x and integer n

Output: The value x^n

if $n = 0$ **then**

return 1

if n is odd **then**

$y = \text{Power}(x, (n - 1)/2)$

return $x \cdot y \cdot y$

else

$y = \text{Power}(x, n/2)$

return $y \cdot y$

Analyzing the Recursive Squaring Method

Algorithm Power(x, n):

Input: A number x and integer $n = 0$

Output: The value x^n

```
if  $n = 0$  then  
    return 1  
if  $n$  is odd then  
     $y = \text{Power}(x, (n - 1)/2)$   
    return  $x \cdot y \cdot y$   
else  
     $y = \text{Power}(x, n/2)$   
    return  $y \cdot y$ 
```

Although there are 2 statements that recursively call Power, only one is executed per stack frame.

Each time we make a recursive call we halve the value of n (roughly).

Thus we make a total of $\log n$ recursive calls. That is, this method runs in $O(\log n)$ time.

Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its **last** step.
- Such a method can easily be converted to an iterative method (which saves on some resources).

Example: Recursively Reversing an Array

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

if $i < j$ **then**

 Swap $A[i]$ and $A[j]$

 ReverseArray($A, i + 1, j - 1$)

return

Example: Iteratively Reversing an Array

Algorithm IterativeReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

while $i < j$ **do**

 Swap $A[i]$ and $A[j]$

$i = i + 1$

$j = j - 1$

return

Defining Arguments for Recursion

- Solving a problem recursively sometimes requires passing additional parameters.
- **ReverseArray** is a good example: although we might initially think of passing only the array **A** as a parameter at the top level, lower levels need to know where in the array they are operating.
- Thus the recursive interface is **ReverseArray(A, i, j)**.
- We then invoke the method at the highest level with the message **ReverseArray(A, 1, n)**.

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Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example 1: **The Fibonacci Sequence**

The Fibonacci Sequence

- Fibonacci numbers are defined recursively:

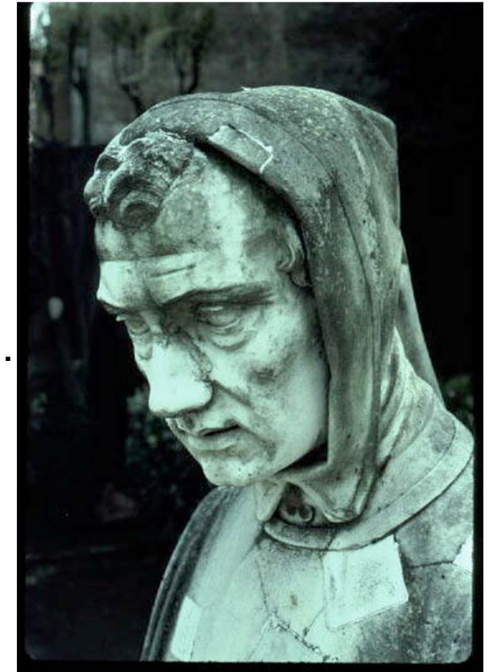
$$F_0 = 0$$

$$F_1 = 1$$

$$F_i = F_{i-1} + F_{i-2} \quad \text{for } i > 1.$$

The ratio F_i / F_{i-1} converges to $\varphi = \frac{1+\sqrt{5}}{2} = 1.61803398874989\dots$

(The “**Golden Ratio**”)

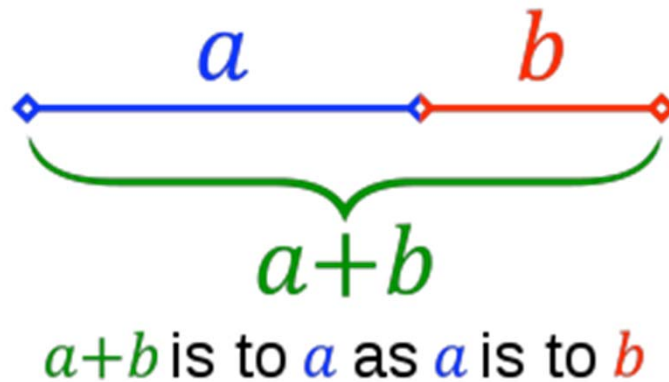


Fibonacci (c. 1170 - c. 1250)
(aka Leonardo of Pisa)

The Golden Ratio

- Two quantities are in the **golden ratio** if the ratio of the sum of the quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one.

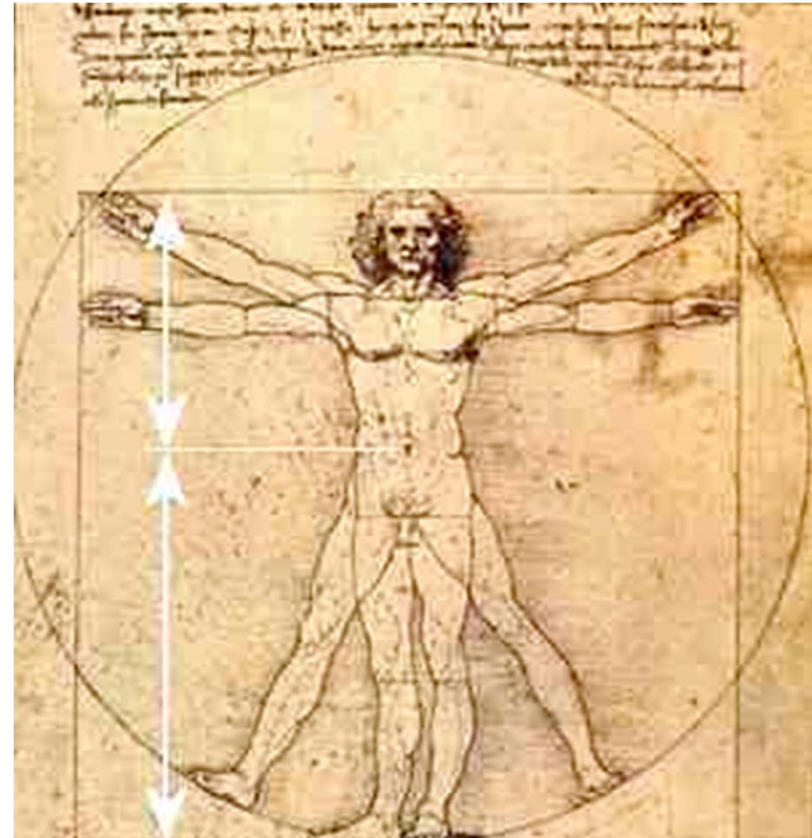
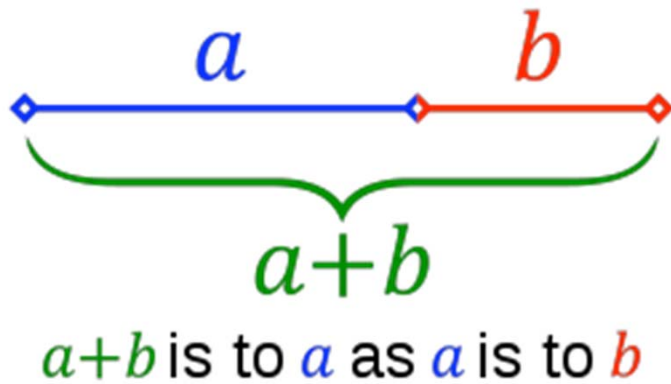
φ is the unique positive solution to $\varphi = \frac{a+b}{a} = \frac{a}{b}$.



The Golden Ratio



The Parthenon



Leonardo

Computing Fibonacci Numbers

$$F_0 = 0$$

$$F_1 = 1$$

$$F_i = F_{i-1} + F_{i-2} \quad \text{for } i > 1.$$

- A recursive algorithm (first attempt):

Algorithm BinaryFib(k):

Input: Positive integer k

Output: The k th Fibonacci number F_k

if $k < 2$ **then**

return k

else

return BinaryFib($k - 1$) + BinaryFib($k - 2$)

Analyzing the Binary Recursion Fibonacci Algorithm

- Let n_k denote number of recursive calls made by BinaryFib(k).
Then
 - $n_0 = 1$
 - $n_1 = 1$
 - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
 - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
 - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
 - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
 - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
 - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
 - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67.$
- Note that n_k more than doubles for every other value of n_k . That is, $n_k > 2^{k/2}$. It increases exponentially!

A Better Fibonacci Algorithm

- Use **linear** recursion instead:

Algorithm LinearFibonacci(k):

Input: A positive integer k

Output: Pair of Fibonacci numbers (F_k, F_{k-1})

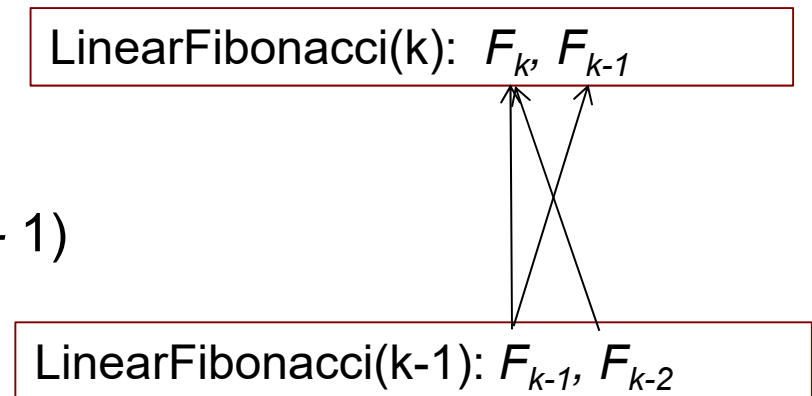
if $k = 1$ **then**

return $(k, 0)$

else

$(i, j) = \text{LinearFibonacci}(k - 1)$

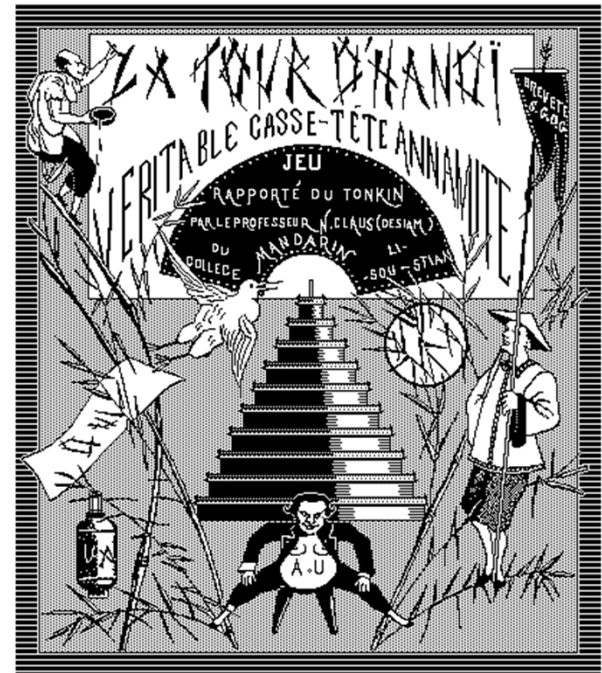
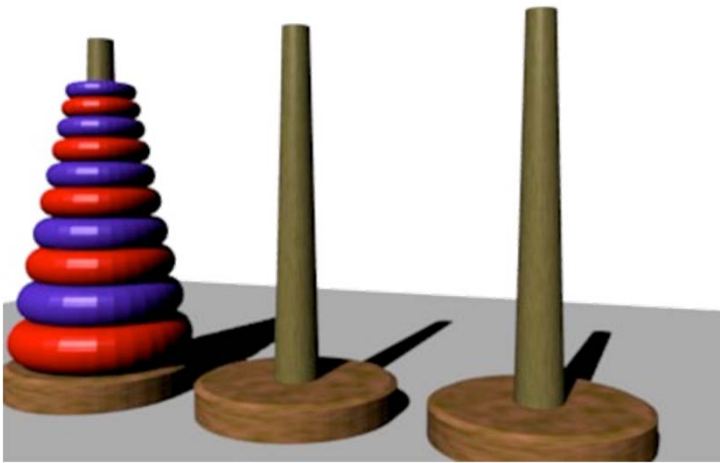
return $(i + j, i)$



- Runs in **$O(k)$** time.

Binary Recursion

- Second Example: **The Tower of Hanoi**



Example

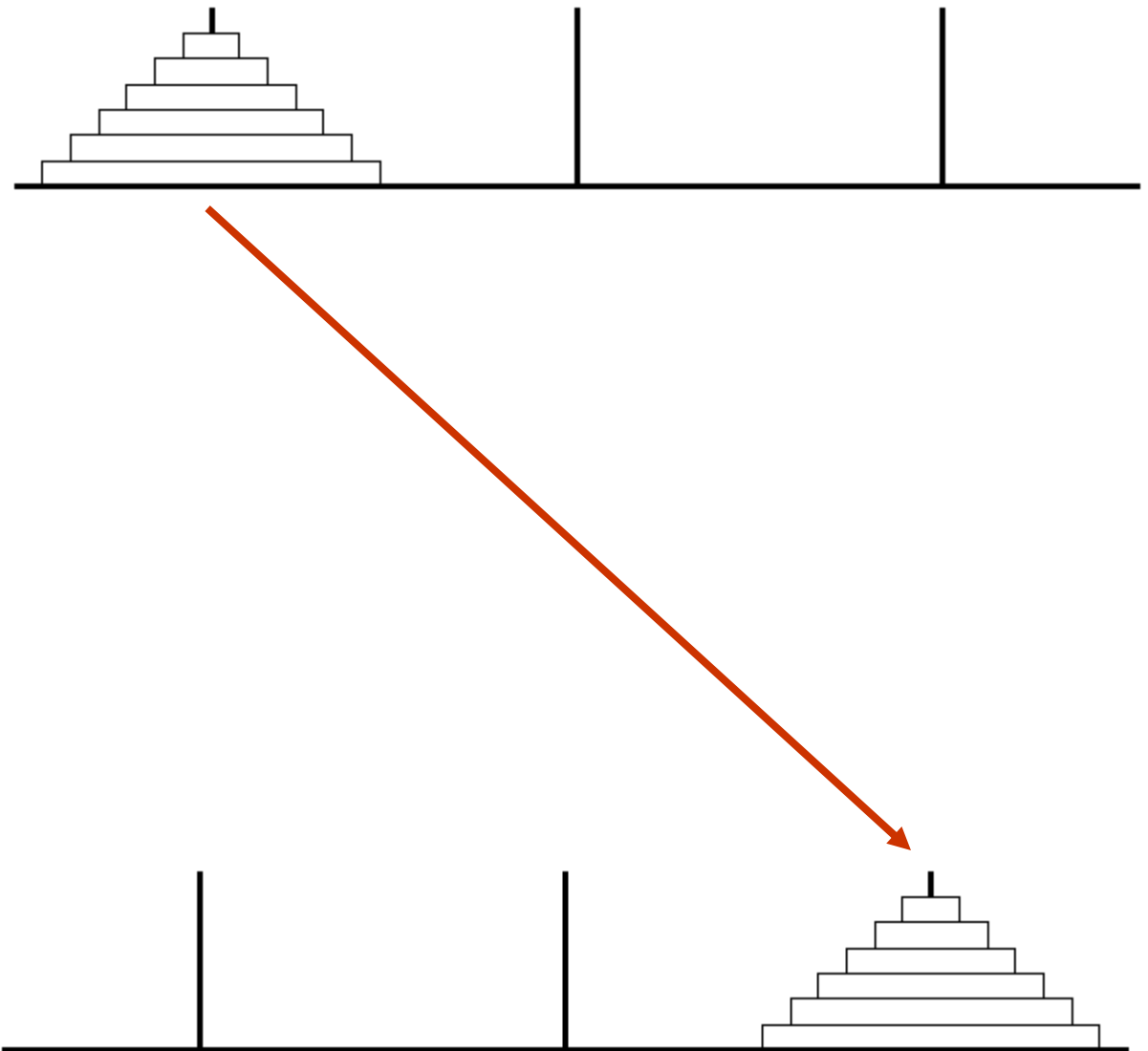
Tower of Hanoi



This job of mine
is a bit daunting.
Where do I start?

And I am lazy.

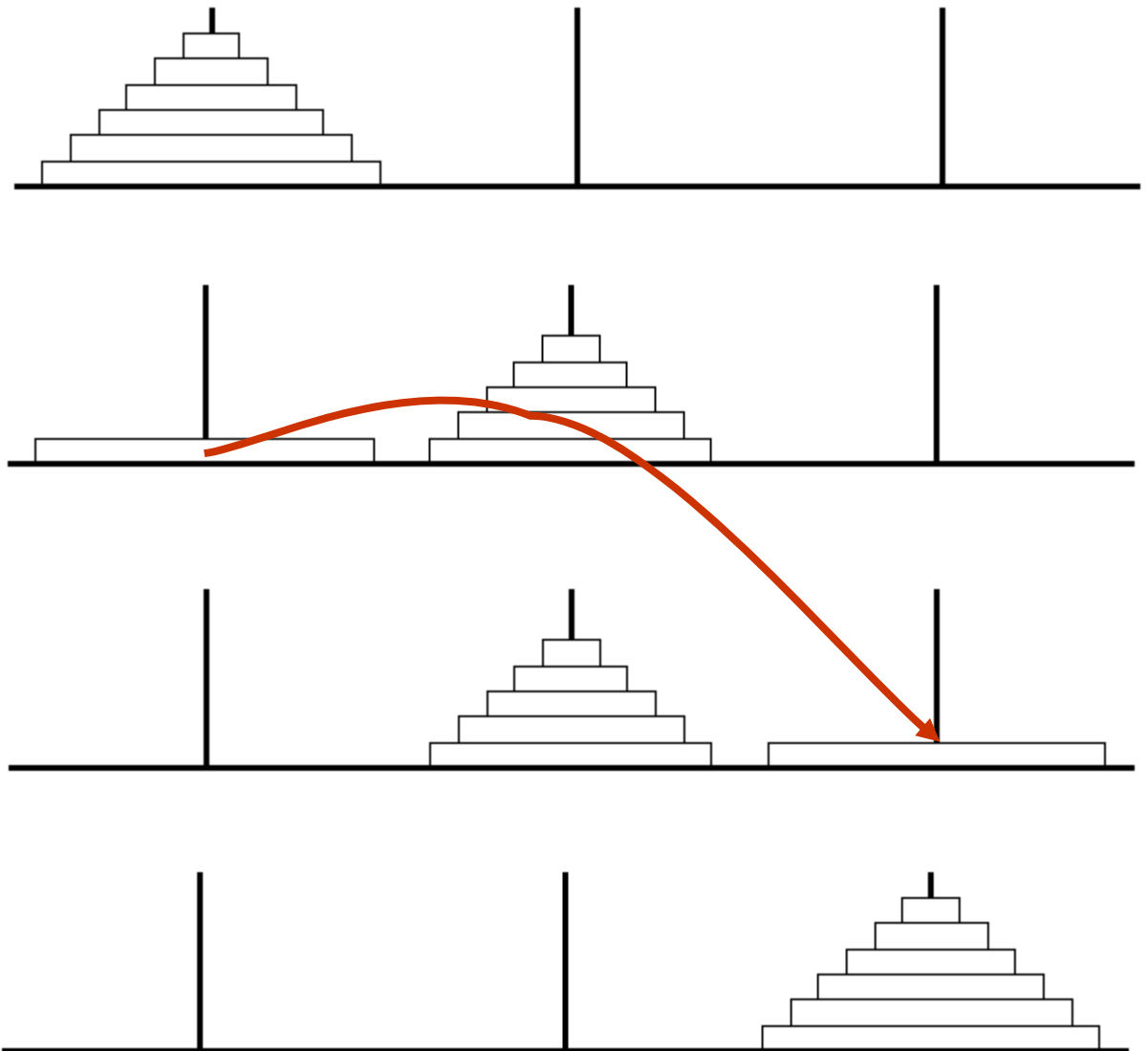
Example from J. Edmonds – Thanks Jeff!



Tower of Hanoi



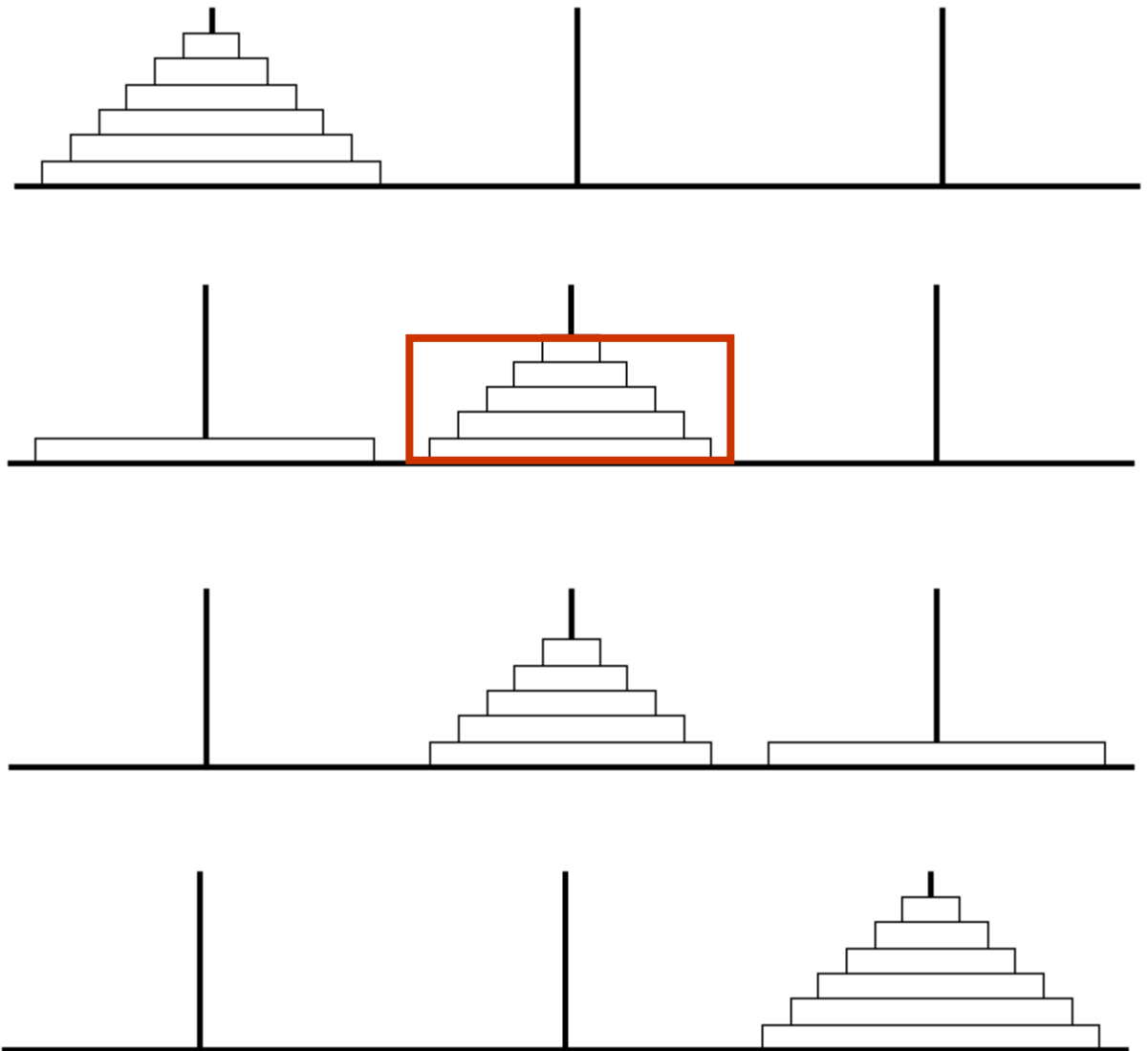
At some point,
the biggest disk
moves.
I will do that job.



Tower of Hanoi



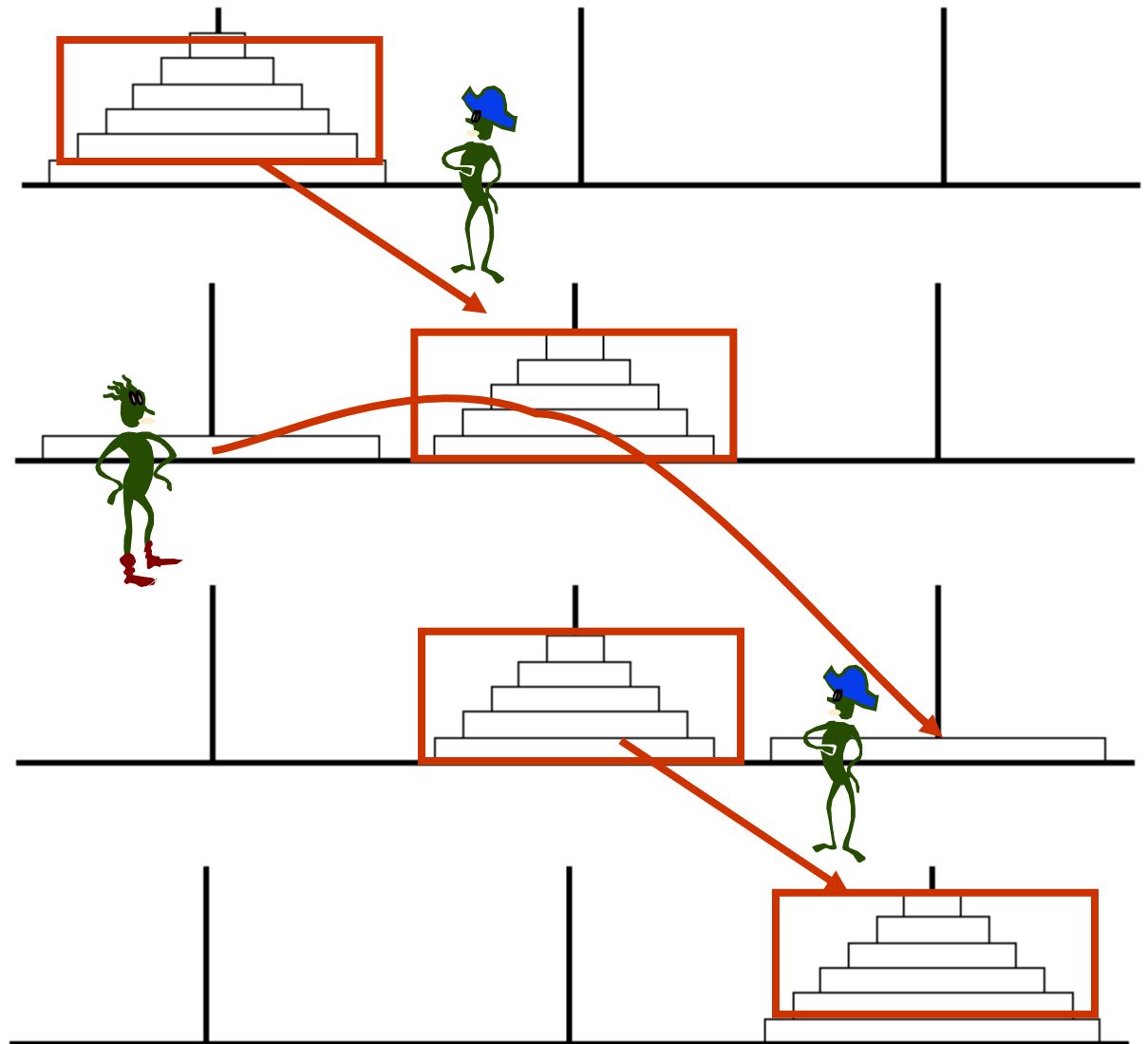
To do this,
the other disks
must be in the
middle.



Tower of Hanoi



How will these
move?
I will get a
friend to do it.
And another to
move these.
I only move the
big disk.



Tower of Hanoi

Code:

```
algorithm TowersOfHanoi( $n$ ,  $source$ ,  $destination$ ,  $spare$ )
```

```
  <pre-cond>: The  $n$  smallest disks are on  $pole_{source}$ .
```

```
  <post-cond>: They are moved to  $pole_{destination}$ .
```

```
begin
```

```
  if( $n = 1$ )
```

```
    Move the single disk from  $pole_{source}$  to  $pole_{destination}$ .
```

```
  else
```

```
    TowersOfHanoi( $n - 1$ ,  $source$ ,  $spare$ ,  $destination$ )
```

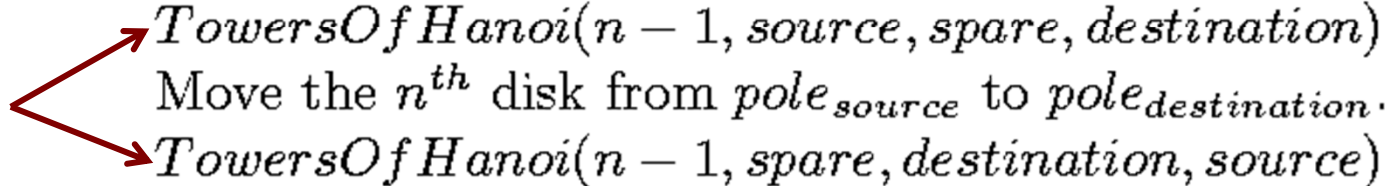
```
    Move the  $n^{th}$  disk from  $pole_{source}$  to  $pole_{destination}$ .
```

```
    TowersOfHanoi( $n - 1$ ,  $spare$ ,  $destination$ ,  $source$ )
```

```
  end if
```

```
end algorithm
```

**2 recursive
calls!**



Tower of Hanoi

Code:

```
algorithm TowersOfHanoi(n, source, destination, spare)
```

```
  <pre-cond>: The n smallest disks are on polesource.
```

```
  <post-cond>: They are moved to poledestination.
```

```
  begin
```

```
    if(n = 1)
```

```
      Move the single disk from polesource to poledestination.
```

```
    else
```

```
      TowersOfHanoi(n - 1, source, spare, destination)
```

```
      Move the nth disk from polesource to poledestination.
```

```
      TowersOfHanoi(n - 1, spare, destination, source)
```

```
    end if
```

```
  end algorithm
```

Time:

$$T(1) = 1,$$

$$T(n) = 1 + 2T(n-1) \approx 2T(n-1)$$

$$\approx 2(2T(n-2)) \approx 4T(n-2)$$

$$\approx 4(2T(n-3)) \approx 8T(n-3)$$

$$\approx 2^i T(n-i)$$

$$\approx 2^n$$

Binary Recursion: Summary

- Binary recursion spawns an exponential number of recursive calls.
- If the inputs are only declining **arithmetically** (e.g., $n-1$, $n-2$, ...) the result will be an exponential running time.
- In order to use binary recursion, the input must be declining **geometrically** (e.g., $n/2$, $n/4$, ...).
- We will see efficient examples of binary recursion with geometrically declining inputs when we discuss **heaps** and **sorting**.

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The Overhead Costs of Recursion

- Many problems are naturally defined recursively.
- This can lead to simple, elegant code.
- However, recursive solutions entail a **cost in time and memory**: each recursive call requires that the current process state (variables, program counter) be **pushed** onto the system stack, and **popped** once the recursion unwinds.
- This typically affects the running time **constants**, but **not** the **asymptotic time complexity** (e.g., $O(n)$, $O(n^2)$ etc.)
- Thus **recursive solutions may still be preferred** unless there are very strict time/memory constraints.

The “Curse” in Recursion: Errors to Avoid

```
// recursive factorial function  
public static int recursiveFactorial(int n) {  
    return n * recursiveFactorial(n- 1);  
}
```

- **There must be a base condition: the recursion must ground out!**

The “Curse” in Recursion: Errors to Avoid

```
// recursive factorial function
public static int recursiveFactorial(int n) {
    if (n == 0) return recursiveFactorial(n);    // base case
    else return n * recursiveFactorial(n- 1);    // recursive case
}
```

- **The base condition must not involve more recursion!**

The “Curse” in Recursion: Errors to Avoid

```
// recursive factorial function
public static int recursiveFactorial(int n) {
    if (n == 0) return 1;    // base case
    else return (n - 1) * recursiveFactorial(n); // recursive case
}
```

- The input **must be converging** toward the base condition!

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