Linear Temporal Logic

EECS 4315

www.eecs.yorku.ca/course/4315/

Linear temporal logic

Linear temporal logic (LTL) is a logic to reason about systems with nondeterminism.

The logic was introduced by Amir Pnueli.

A. Pnueli. The temporal logic of programs. In *Proceedings of the 18th IEEE Symposium on Foundations of Computer Science*, pages 46–67. Providence, RI, USA, October/November 1977. IEEE.

Amir Pnueli (1941–2009)

- Recipient of the Turing Award (1996)
- Recipient of the Israel prize (2000)
- Foreign Associate of the U.S. National Academy of Engineering (1999)
- Fellow of the Association for Computing Machinery (2007)

Source: David Monniaux

Linear temporal logic

Definition

LTL is defined by the grammar

$$f ::= a \mid f \land f \mid \neg f \mid \bigcirc f \mid f \,\mathsf{U}\, f$$

where a is an atomic proposition.

An atomic proposition represents a basic property (such as the value of a particular variable being even or a particular method being invoked).

LTL

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$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \, \mathsf{U} \, f$$

Question

Is $a \wedge \neg b$ is an LTL formula?

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Answer

No.

Definition

LTL is defined by the grammar

$$f ::= a \mid f \wedge f \mid \neg f \mid \bigcirc f \mid f \,\mathsf{U}\, f$$

Question

Is $a \wedge \neg(\bigcirc b \,\mathsf{U}\, c)$ is an LTL formula?

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Is $a \wedge \neg(\bigcirc b \mathbin{\mathsf{U}} c)$ is an LTL formula?

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Given an execution path p, does it satisfy a particular LTL formula f?

An atomic proposition a is satisfied if a holds in the initial state of the execution path.

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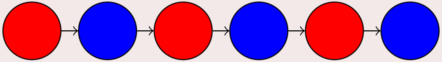

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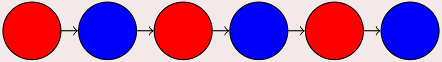

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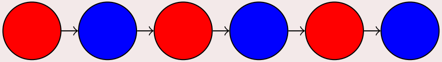

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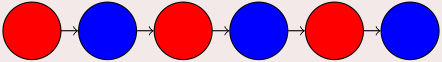

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Given an execution path p, does it satisfy a particular LTL formula f?

The LTL formula $\bigcirc a$ (pronounced as next a) is satisfied if a holds in the next state of the execution path (that is, the second state).

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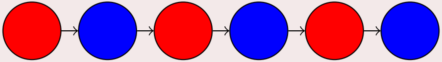

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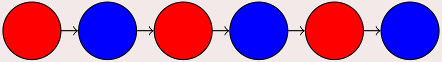

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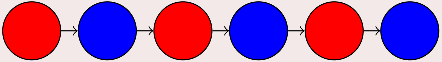

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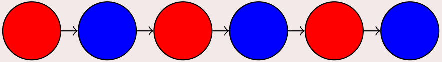

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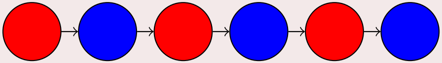

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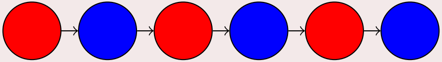

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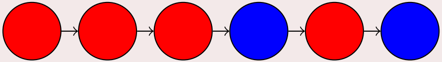

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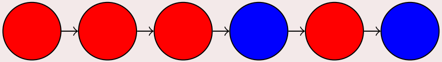

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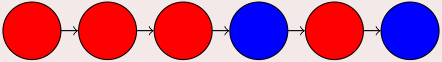

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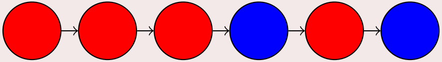

satisfy the atomic proposition blue U red?

Answer

Yes![a]

[a] All states before the first red state are blue.

Syntactic sugar

As usual

$$\begin{aligned} \text{true} &= a \vee \neg a \\ f \vee g &= \neg(\neg f \wedge \neg g) \\ f \Rightarrow g &= \neg f \vee g \end{aligned}$$

Also

$$\Diamond f = \text{true} \,\mathrm{U}\, f \qquad \text{(eventually } f\text{)}$$

$$\Box f = \neg \Diamond \neg f \qquad \text{(always } f\text{)}$$

Alternative syntax

$$\mathsf{X}f : \bigcirc f$$

$$\mathsf{F}f : \Diamond f$$

$$\mathsf{G}f : \Box f$$

We introduce two basic tense operators, F and G.

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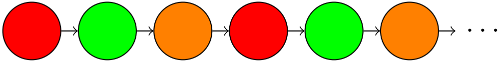

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Which LTL formula expresses "initially the light is red and next it becomes green."

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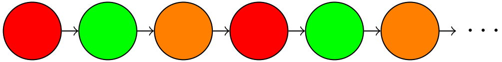

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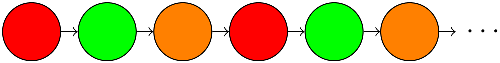

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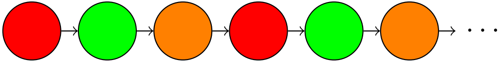

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$\Diamond$amber

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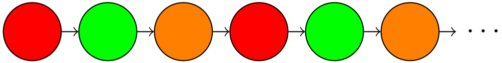

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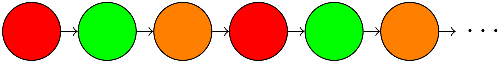

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Answer

$\Box\Diamond$red

Question

What does the formula $\Box(\text{green} \Rightarrow \neg \bigcirc \text{red})$ express?

LTL

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What does the formula $\Box(\text{green} \Rightarrow \neg \bigcirc \text{red})$ express?

Answer

"Once green, the light cannot become red immediately."

State space diagram

Question

Draw the state space diagram of a model of a traffic light. Label (with colours) the states.

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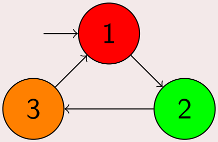

Transition system

Definition

A transition system is a tuple $\langle S, L, I, \rightarrow, \ell \rangle$ consisting of

- a set S of states,
- a set L of labels,
- a set $I \subseteq S$ of initial states,
- a transition relation $\rightarrow \subseteq S \times S$ such that for all $s \in S$ there exists $t \in S$ such that $s \rightarrow t$, and
- a labelling function $\ell : S \rightarrow 2^L$.

2^L denotes the set of subsets of L.

Powerset

Question

What is $2^{\{1,2,3\}}$?

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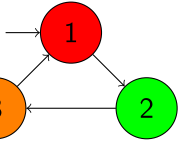
Answer

$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Transition system

Question

Formally define the transition system modelling a traffic light.



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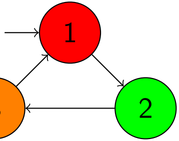
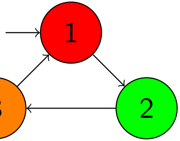


Answer

$\langle \{1,2,3\}, \{\text{red}, \text{green}, \text{amber}\}, \{1\}, \{1 \to 2, 2 \to 3, 3 \to 1\}, \{1 \mapsto \{\text{red}\}, 2 \mapsto \{\text{green}\}, 3 \mapsto \{\text{amber}\}\}\rangle$

Execution paths



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A path is an infinite sequence of states. $Paths(s)$ is the set of path starting in state s.

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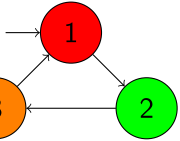

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What is $Paths(2)$?

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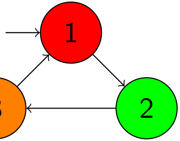

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$Paths(2) = \{231231231\ldots\}$

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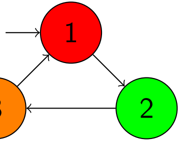

Definition

Let $p \in Paths(s)$ and $n \geq 0$. Then $p[n]$ is the $(n+1)$th state of the path p.

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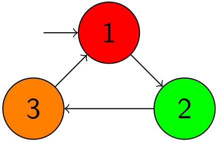

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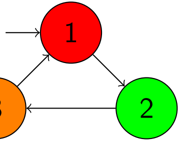

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$p[3] = 1$.

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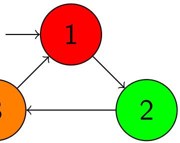

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Let $p \in Paths(s)$ and $n \geq 0$. Then $p[n..]$ is the suffix starting with the $(n+1)$th state of the path p.

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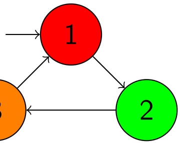

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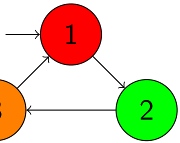

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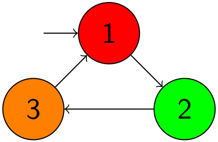
$p[2..] = 312312\ldots$.

Semantics of LTL



$p \models f$ denotes that path p satisfies LTL formula f.

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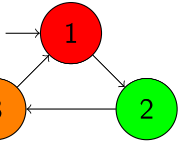

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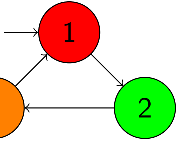

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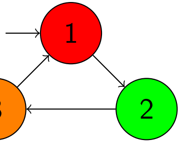

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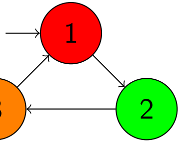

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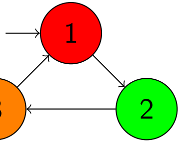

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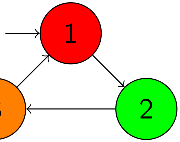

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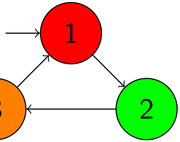
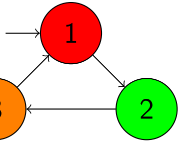


$p \models f$ denotes that path p satisfies LTL formula f.

Question

$123123\ldots \models \neg$green?

Semantics of LTL



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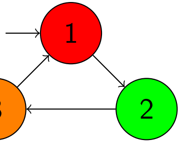

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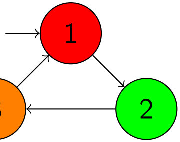

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Question

$123123\ldots \models$ red U green?

Answer

Yes.

Semantics of LTL

Definition

$$
\begin{aligned}
p &\models a \text{ iff } a \in \ell(p[0]) \\
p &\models f \wedge g \text{ iff } p \models f \wedge p \models g \\
p &\models \neg f \text{ iff } p \not\models f \\
p &\models \bigcirc f \text{ iff } p[1..] \models f \\
p &\models f \,\mathsf{U}\, g \text{ iff } \exists i \geq 0 : p[i..] \models g \wedge \forall 0 \leq j < i : p[j..] \models f
\end{aligned}
$$

Semantics of LTL

Question

How can we express $p \models \Diamond f$ in terms of $\cdots \models f$?

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How can we express $p \models \Diamond f$ in terms of $\cdots \models f$?

Answer

$$
\begin{aligned}
p &\models \Diamond f \\
&\text{iff } p \models \text{true} \,\mathsf{U}\, f \\
&\text{iff } \exists i \geq 0 : p[i..] \models f \wedge \forall 0 \leq j < i : p[j..] \models \text{true} \\
&\text{iff } \exists i \geq 0 : p[i..] \models f
\end{aligned}
$$

Question

How can we express $p \models \Box f$ in terms of $\cdots \models f$?

Semantcs of LTL

Question

How can we express $p \models \Box f$ in terms of $\cdots \models f$?

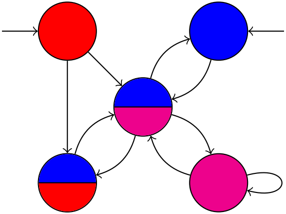
Answer

$$
\begin{aligned}
& p \models \Box f \\
& \quad \text{iff } p \models \neg\Diamond\neg f \\
& \quad \text{iff } \neg(\exists i \geq 0 : p[i..] \models \neg f) \\
& \quad \text{iff } \forall i \geq 0 : p[i..] \models f
\end{aligned}
$$

Semantcs of LTL

Let $TS = \langle S, L, I, \rightarrow, \ell \rangle$ be a transition system. Then

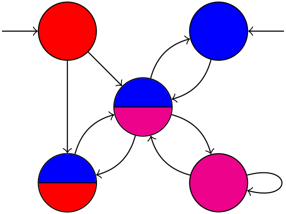
$$TS \models f \text{ iff } \forall s \in I : \forall p \in Paths(s) : p \models f$$



Question

$TS \models \Diamond \text{magenta}$?

Semantcs of LTL



Question

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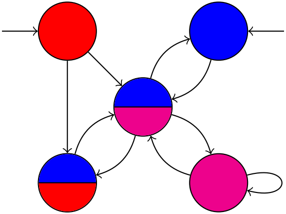
Answer

Yes.

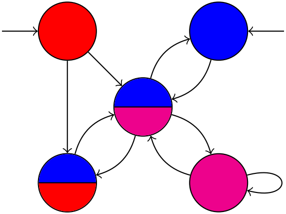
Semantcs of LTL



Question

$TS \models \Box\Diamond$blue?

Semantcs of LTL

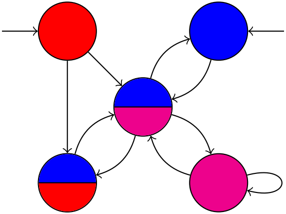


Question

$TS \models \Box\Diamond \text{blue}$?

Answer

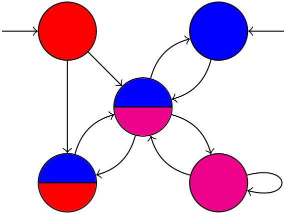
No.



Question

$TS \models \Box(\neg \text{blue} \Rightarrow \bigcirc(\text{magenta} \vee \text{red}))$?

Semantcs of LTL



Question

$TS \models \Box(\neg \text{blue} \Rightarrow \bigcirc(\text{magenta} \vee \text{red}))$?

Answer

Yes.

LTL and JPF

Since the "size" of a transition in JPF can be influenced by the property `vm.max_transition_length`, LTL's next operator $\bigcirc$ is not well-defined in the context of JPF.

Therefore, in the context of JPF we may want to consider the logic defined by the grammar

$$f ::= a \mid f \wedge f \mid \neg f \mid f \: \mathsf{U} \: f$$

where a is an atomic proposition.

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The extensions bitbucket.org/petercipov/jpf-ltl and bitbucket.org/michelelombardi/jpf-ltl of JPF support LTL, but neither is stable.

Equivalence

Definition

The LTL formulas f and g are equivalent, denoted $f \equiv g$, if for all transition systems TS,

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Exercise

Are the following formulas equivalent? Either provide a proof or a counter example.

(a) $\Diamond(f \wedge g) \equiv \Diamond f \wedge \Diamond g$?

(b) $\Diamond \bigcirc f \equiv \bigcirc \Diamond f$?

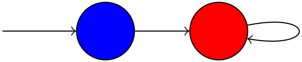
More practice questions can be found in the textbook.

$\Diamond(f \wedge g) \not\equiv \Diamond f \wedge \Diamond g$

For the counter example we provide two ingredients:

- a transition system, and
- LTL formulas for f and g.

Consider the following transition system TS.



Let f = blue and g = red. Then $TS \models \Diamond f \wedge \Diamond g$ but $TS \not\models \Diamond(f \wedge g)$.

Equivalence

$\Diamond \bigcirc f \equiv \bigcirc \Diamond f$

Proof: Let TS be a transition system. Let $s \in I$ and $p \in Paths(s)$. Then

$$
\begin{aligned}
& p \models \Diamond \bigcirc f \\
& \text{iff } \exists i \geq 0 : p[i..] \models \bigcirc f \\
& \text{iff } \exists i \geq 0 : p[i..][1..] \models f \\
& \text{iff } \exists i \geq 0 : p[(i+1)..] \models f \\
& \text{iff } \exists i \geq 0 : p[1..][i..] \models f \\
& \text{iff } p[1..] \models \Diamond f \\
& \text{iff } p \models \bigcirc \Diamond f
\end{aligned}
$$

Course evaluation

The course evaluation for this course can now be completed at
`https://courseevaluations.yorku.ca`

I would really appreciate it if you would take the time to complete the course evaluation. Your feedback allows me to improve the course for future students.

If at least 80% of the students in the course (that is, 12) complete the evaluation, I will bring cup cakes for the last lecture.