Partially Ordered Sets and Fixed Points

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Partially ordered sets and fixed points play a role in variety of areas within theoretical computer science. For example, they are used in the semantics of programming languages to model iterative and recursive programming constructs and in theory of databases to order the answers to a query. Partially ordered sets and fixed points also play a role in the field of model checking. Below, we will introduce those notions that are relevant in that setting. For a more general introduction to partially ordered sets and fixed points we refer the reader to, for example, [DP02].

Definition 1 *A* partially ordered set *is a tuple* $\langle A, \sqsubseteq \rangle$ *consisting of*

- *a set* A *and*
- *a relation* $\sqsubseteq \, \subseteq A \times A$ *satisfying for all* a, b, *and* $c \in A$,
  1. $a \sqsubseteq a$,
  2. *if* $a \sqsubseteq b$ *and* $b \sqsubseteq a$ *then* $a = b$, *and*
  3. *if* $a \sqsubseteq b$ *and* $b \sqsubseteq c$ *then* $a \sqsubseteq c$.

The above three conditions are known as reflexivity, antisymmetry and transitivity, respectively. A partially ordered set is also known as a poset.

Example 2

1. *The set* $\{a, b, c\}$ *together with the relation* $\{(a, a), (a, b), (a, c), (b, b), (c, c)\}$ *forms a partially ordered set.*
2. *The unit interval* $[0, 1]$ *together the relation* $\leq$ *is a partially ordered set.*
3. *Let* S *be the set. Then the set* 2^S *of subsets of* S *together with the inclusion relation* $\subseteq$ *form a partially ordered set.*

Least upper bounds and their duals, greatest lower bounds, can be used to define fixed points.

Definition 3 *Let* $\langle A, \sqsubseteq \rangle$ *be a partially ordered set and* $B \subseteq A$.

- $a \in A$ *is an* upper bound *of* B *iff* $b \sqsubseteq a$ *for all* $b \in B$.
- $a \in A$ *is a* least upper bound *of* B *iff*
  - a *is an upper bound of* B, *and*

- *for all $a' \in A$, if a' is an upper bound of B then $a \sqsubseteq a'$.*

- *$a \in A$ is a* lower bound *of B iff $a \sqsubseteq b$ for all $b \in B$.*
- *$a \in A$ is a* greatest lower bound *of B iff*
  - *a is a lower bound of B, and*
  - *for all $a' \in A$, if a' is a lower bound of B then $a' \sqsubseteq a$.*

If a set B has a least upper bound, then it is unique, and we denote it by $\sqcup B$. Similarly, if a set B has a greatest lower bound, then it is unique as well, and we denote it by $\sqcap B$.

Example 4

1. *Consider the partially ordered set defined in Example 2.1. The greatest lower bound of $\{b, c\}$ is a. This set does not have a least upper bound.*
2. *Consider the partially ordered set defined in Example 2.2. The greatest lower bound of $(0, 0.5)$ is 0 and its least upper bound is 0.5.*
3. *Consider the partially ordered set defined in Example 2.3. Then intersection and union correspond to $\sqcap$ and $\sqcup$, respectively.*

Definition 5 *Consider the function $F : A \to A$. Then*

- *$a \in A$ is a* fixed point *of F iff $F(a) = a$,*
- *$a \in A$ is a* pre-fixed point *of F iff $F(a) \sqsubseteq a$, and*
- *$a \in A$ is a* post-fixed point *of F iff $a \sqsubseteq F(a)$.*

Obviously, not every function has a fixed point. We will restrict ourselves to a particular class of functions and a particular class of partial orders.

Definition 6 *Let $\langle A, \sqsubseteq \rangle$ be a partial order. A function $F : A \to A$ is* monotone *iff for all a, $b \in A$, if $a \sqsubseteq b$ then $F(a) \sqsubseteq F(b)$.*

Example 7

1. *Consider the partially ordered set defined in Example 2.1. The function $F : \{a, b, c\} \to \{a, b, c\}$ defined by $F(a) = a$, $F(b) = a$ and $F(c) = c$ is monotone.*
2. *Consider the partially ordered set defined in Example 2.2. The function $F : [0, 1] \to [0, 1]$ defined by $F(r) = \frac{r}{2}$ is monotone and has 0 as its only fixed point and its only post-fixed point. Each $r \in [0, 1]$ is a pre-fixed point of F.*
3. *Consider the partially ordered set defined in Example 2.3. Let $X \subseteq S$. The function $F : 2^S \to 2^S$ defined by $F(Y) = Y \cap X$ is monotone. Each subset of X is a fixed point of F and a post-fixed point of F. Each subset of S is a pre-fixed point of F.*

Definition 8 *A partially ordered set $\langle A, \sqsubseteq \rangle$ is a* complete lattice *if every subset of A has a least upper bound and a greatest lower bound.*

Example 9 *The partially ordered sets defined in 2 and 3 of Example 2 are complete lattices but the one defined in 1 is not.*

The Knaster-Tarski theorem, named after Bronislaw Knaster and Alfred Tarski, states that the set of fixed points of a monotone function on a complete lattice is itself also a complete lattice [Kna28, Tar55]. Since a complete lattice has a least and a greatest element, we can conclude from the Knaster-Tarski theorem the following result.

Theorem 10 *Let $\langle A, \sqsubseteq \rangle$ be a complete lattice. If the function $F : A \to A$ is monotone, then it has a least fixed point (which is the least pre-fixed point) and a greatest fixed point (which is the greatest post-fixed point).*

References

[DP02] B.A. Davey and H.A. Priestley. *Lattices and order*. Cambridge University Press, second edition, 2002.

[Kna28] B. Knaster. Un théorème sur les fonctions d'ensembles. *Annales de la Société Polonaise de Mathématique*, 6:133–134, 1928.

[Tar55] A. Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309, 1955.