Loop Antenna Equations

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***Abstract*— A brief discussion of a loop antenna in empty space that also serves to catalog some important antenna relations.**
Revised: June 6, 2007

I. Loop Radiation

An excellent discussion of loop antenna physics can be found in McDonald's webnotes [1]. Therein the general expressions for the radiation properties of any parallelogram loop are considered. Assuming a square loop a meters on a side we can calculate for the radiation intensity (in Watts/steradian) [2]

$$U(\theta,\phi)=\frac{2I_0^2}{\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\sin^2(\frac{\pi a}{\lambda}\sin\theta\cos\phi)\sin^2(\frac{\pi a}{\lambda}\sin\theta\sin\phi)}{\sin^2\theta\sin^2 2\phi} \tag{1}$$

where I_0 is the amplitude of the sinusoidal current flowing in the loop, λ is the wavelength of the radiation signal, θ is the elevation angle (with respect to the plane of the loop), ϕ is the azimuth angle (in the plane of the loop), μ_0 is the permeability of vacuum, and ϵ_0 is the permittivity of vacuum.

The total amount of power absorbed/emitted by this antenna is

$$P_{rad}=\frac{I_0^2 R_r}{2} \tag{2}$$

where R_r is the radiation resistance. Integrating the reference power density over the spherical solid angle results in the total radiation power, that is

$$P_{rad}=\int_\Omega U(\theta,\phi)\mathrm{d}\Omega \tag{3}$$

so that

$$R_r=\frac{4}{\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\int_\Omega\frac{\sin^2(\frac{\pi a}{\lambda}\sin\theta\cos\phi)\sin^2(\frac{\pi a}{\lambda}\sin\theta\sin\phi)}{\sin^2\theta\sin^2 2\phi}\mathrm{d}\Omega. \tag{4}$$

For comparison the radiation resistance of a "small" loop antenna (i.e. dimensions $\ll\lambda$) without ground plane is [3]

$$R_r=3.12\times10^4\left(\frac{NS}{\lambda^2}\right)^2=3.12\times10^4\left(\frac{N(l_1\cdot l_2)}{\lambda^2}\right)^2 \quad [\Omega] \tag{5}$$

where N is the number of loops, S the loop area, and l_1 and l_2 the antenna dimensions in the case of a square loop. Ideally the antenna dimensions will conform to those of the integrated technology around which the loop is wrapped [4]. This means that we can expect l_1 and l_2 to range from 1 mm or less up to 15 mm — probably a rather aggressive value give that microprocessors typically range about 10 mm on a side. If we consider possible frequencies of operation up to say 60 GHz, we must resort to a more involved account of the antenna properties, as presented above, since the wavelength at these frequencies (5 mm at 60 GHz) is on the same order as the antenna dimensions.

If we know enough about the material details of the loop antenna we can calculate the loss resistance, R_l, and hence the antenna efficiency,

$$\eta=\frac{R_r}{R_r+R_l}. \tag{6}$$

This factor has a place in the consideration of the antennas effective aperture area (the aperture), A_e. We can think of A_e as the area through which the radiative flux power appearing at the antenna's terminals had to pass (to reach the antenna's terminals). From [2] we recall that

$$A_e(\theta,\phi)=\lambda^2\frac{U(\theta,\phi)}{P_{inj}}=\eta\lambda^2\frac{U(\theta,\phi)}{P_{rad}} \tag{7}$$

where P_{inj} is the power appearing at the antenna's terminals (i.e. either coming from the transmitter circuit in transmit mode or available to the receiver in receive mode). Clearly the lower the carrier frequency the greater the antenna's aperture.

Substituting Eq. (6) into Eq. (7) we get

$$A_e(\theta,\phi)=\frac{2\lambda^2}{R_r+R_l}\frac{U(\theta,\phi)}{I_0^2}. \tag{8}$$

Two other important antenna measures are the gain $G(\theta,\phi)$ and directivity $D(\theta,\phi)$. The key relationships for these variables are

$$G(\theta,\phi)=\frac{4\pi}{\lambda^2}A_e(\theta,\phi)=\frac{4\pi U(\theta,\phi)}{P_{inj}}. \tag{9}$$

As noted in [2] the when referring to the gain without any arguments, that is, as just G, we are implicitly referring to the maximum antenna gain. The directivity is simply the gain corrected for the antenna efficiency

$$D(\theta,\phi)=\frac{G(\theta,\phi)}{\eta}=\frac{4\pi U(\theta,\phi)}{P_{rad}}. \tag{10}$$

II. Loop Inductance

The inductance of a rectangular loop of wire of radius r and dimensions $l_1\times l_2$ is given by Stutzman [3] as

$$L=\frac{\mu_0}{\pi}\left(l_2\cdot\mathrm{acosh}\frac{l_1}{2r}+l_1\cdot\mathrm{acosh}\frac{l_2}{2r}\right) \tag{11}$$

Since we will be dealing with planar coil designs, a suitable approximation to r for a planar coil of line width, w, is

$$r\approx w/2\pi. \tag{12}$$

The assumption is that we can treat the width of the planar coil as the circumference of circular coil.

Many thanks to the friends of FishLab.

III. Loop Losses

In calculating the loop's resistive losses we start by noting that the skin depth of a metal is given by

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \tag{13}$$

where ω_0 is the (angular) frequency of the signal through the metal with conductivity σ [$\mho/m$]. With this we can approximate the series loss of the coil with

$$R_l \approx \frac{2(l_1 + l_2)}{w \cdot \delta \cdot \sigma} \tag{14}$$

where $1/\sigma$ is multiplied by the total length of the loop (numerator) and divided by the effective cross-sectional area of the loop. Lee [5] (pg. 141) refines this approximation further to

$$R_l \approx \frac{2(l_1 + l_2)}{w \cdot \delta(1 - e^{-t/\delta}) \cdot \sigma}. \tag{15}$$

As noted by Lee, in actual integrated scenarios, the substrate losses and proximity effects would have to be accounted for in an accurate calculation of R_l.

IV. Insights

From Eq. (5) we can see that the radiation resistance of a small loop antenna is proportional to f^4 while R_l (Eq. (15)) is proportional to $\sqrt{f}$, hence our expressions for R_r and R_l combine to predict a rather poor antenna efficiency, η, at low carrier frequencies. Example calculations are given in [4].

References

[1] K. T. McDonald, "A parallelogram loop antenna," May 28 2003, URL: http://physics.princeton.edu/ mcdonald/examples/loopantenna.pdf.

[2] S. Magierowski, "Basics of antenna power transfer," Tech. Rep., June 2 2007.

[3] W. L. Stutzman and G. A. Thiele, *Antenna Theory and Design*, Wiley, 2nd edition, 1997.

[4] S. Magierowski, "Small loop antenna calculations," Tech. Rep., June 6 2007.

[5] T. H. Lee, *The Design of CMOS Radio-Frequency Integrated Circuits*, Cambridge, 2nd edition, 2004.